Finitely forcible limits of graphs and permutations

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Jagiellonian University

join work with R. Glebov, T. Klimošová, D. Král'

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Erdős problem *maximize the number of* C_5 *among all triangle-free graphs* was open almost 30 years. Now is just an easy application of graph limits.

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Every sequence has a convergent subsequence.

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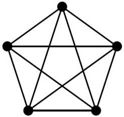

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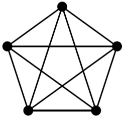

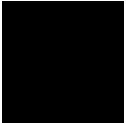

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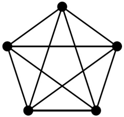

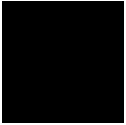

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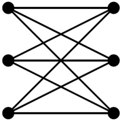

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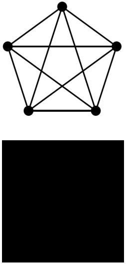

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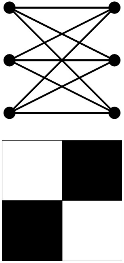

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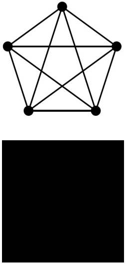

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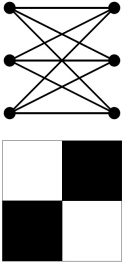

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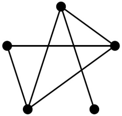

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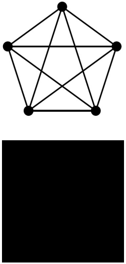

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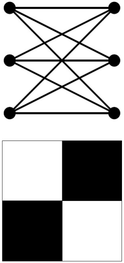

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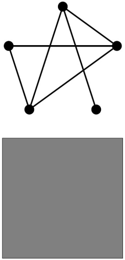

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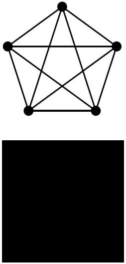

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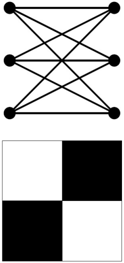

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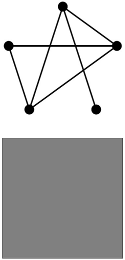

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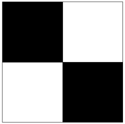

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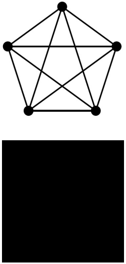

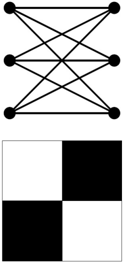

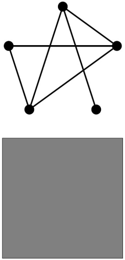

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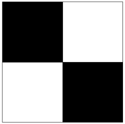

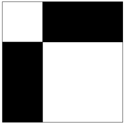

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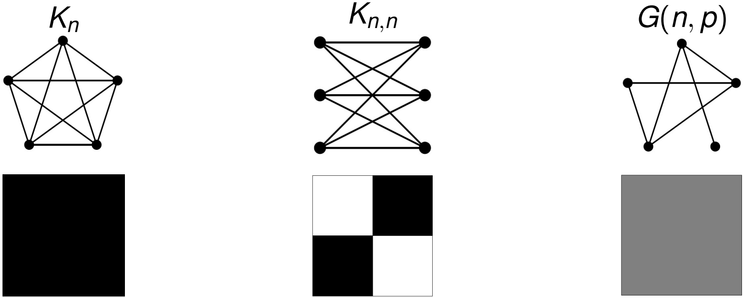

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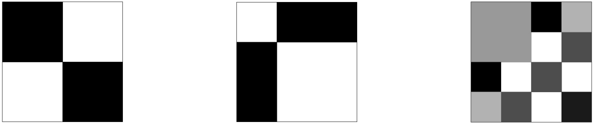

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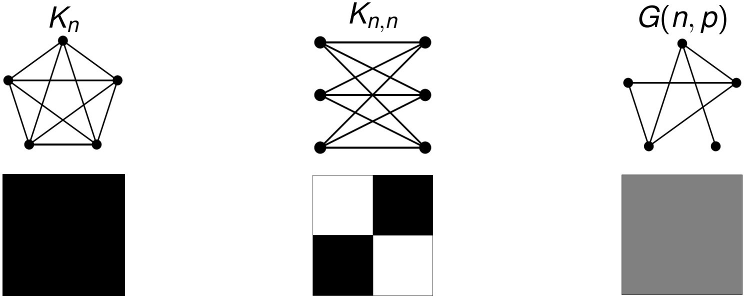

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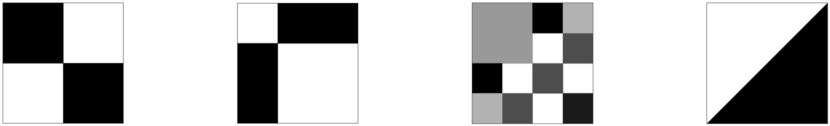

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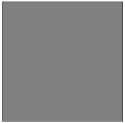

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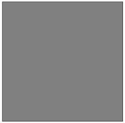

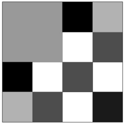

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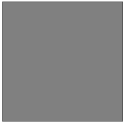
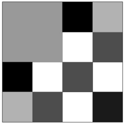
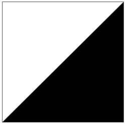

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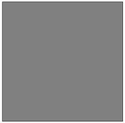

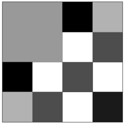

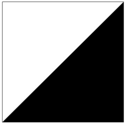

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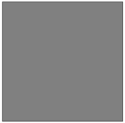
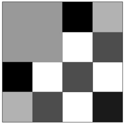
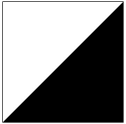

Every finitely forcible graphon is a unique solution to some extremal problem.

Lovász, Szegedy: Every extremal problem has a finitely forcible solution?

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- Glebov, G., Klimošová, Kráľ (2015): Permuton related with union of monotone (or random) permutations is finitely forcible.

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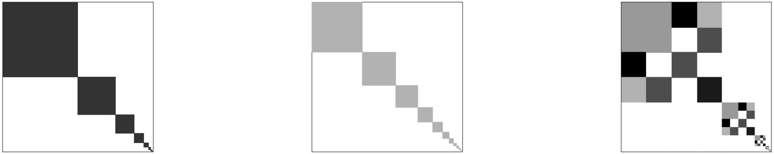

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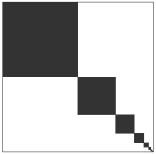

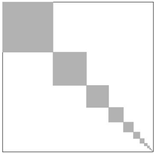

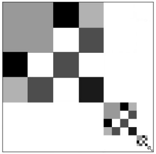

Finite forcibility of permutons is not preserved by the associated graphons.

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R. Glebov, A. Grzesik, T. Klimošová, D. Král, Finitely forcible graphons and permutons, *Journal of Combinatorial Theory. Series B* **110** (2015), 112–135.

Thank you very much for your attention.