Tupled Pregroup Grammars enjoy semantics

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Before we start

Joachim Lambek (1922-2014)
Pregroups and pregroup grammars were introduced in 1999 by Jim Lambek, as a new tool for syntactic analysis of natural languages.
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The formalism of pregroup grammars belongs to the tradition of categorical grammars.

In general they are part of a wide field of mathematical linguistic i.e. the theory of formal grammars and automata with application in computer science, in particular in natural language processing.
Not so formal - how to describe the (natural) language?

Remember: language is a tricky business.
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- word (the element of an alphabet)
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- string of words
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- string of words
- a sentence?
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- type (assigned to a word)
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- a sentence?
- calculation on types
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- type (assigned to a word)
- string of words
- a sentence?
- calculation on types
- answer (Y/N)
Not so formal - how to describe the (natural) language?

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- word (the element of an alphabet)
- type (assigned to a word)
- string of words
- a sentence?
- calculation on types
- answer (Y/N)

- what is the meaning of the sentence?
Not so formal - how to describe the (natural) language?

Remember: language is a tricky business.

- word (the element of an alphabet)
- type (assigned to a word)
- string of words
- a sentence?
- calculation on types
- answer (Y/N)

- what is the meaning of the sentence?
- calculating the meaning???
**Ppregroup** is a structure \((G, \leq, \cdot, \ell, r, 1)\) such that \((G, \leq, \cdot, 1)\) is a partially ordered monoid, and \(\ell\) and \(r\) are unary operations satisfying the following inequalities:

\[
(PRE) \quad a\ell a \leq 1 \leq aa\ell \quad \text{and} \quad aa^r \leq 1 \leq a^ra,
\]

for all \(a \in G\).

\(a\ell\) is called the *left adjoint of* \(a\), whereas \(a^r\) is called the *right adjoint of* \(a\).
Calculus of pregroups
Other approaches
Tupled pregroup grammars
Examples

Introduction
Definition of a pregroup
Example

Example

John
\[ N \]
\[ \pi_3^r s_1 o_4^l \]
Mary
\[ N \] \implies ???
Example

John \[ N \]
likes \[ \pi_3^r \, s_1 \, o_4^l \]
Mary \[ N \] \[ \Rightarrow \] ???

Note: \( N \leq \pi_3 \) and \( N \leq o_4 \)
Example

John \text{ likes} Mary

\begin{align*}
N & \quad \pi_3^r \quad s_1 \quad o_4^l \\
\quad & \quad N \
\end{align*}

⇒ ???

Note: \( N \leq \pi_3 \) and \( N \leq o_4 \)

John \text{ likes} Mary

\begin{align*}
\pi_3 & \quad \pi_3^r \quad s_1 \quad o_4^l \\
\quad & \quad O_4 \
\end{align*}

⇒ s_1
Introduction

Definition of a pregroup

Example

John likes Mary

\[ N \pi_3^r s_1 o_4^l N \Rightarrow ??? \]

Note: \( N \leq \pi_3 \) and \( N \leq o_4 \)

John likes Mary

\[ \pi_3 \pi_3^r s_1 o_4^l o_4 \Rightarrow s_1 \]

John likes Mary.

\[ \pi_3 \pi_3^r s_1 o_4^l N \Rightarrow s_1 \]
Other approaches

pregroups with modalities
(M.Fadda, A.Kiślak-Malinowska)
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pregroups with metarules for word order
(C. Casadio, A. Kiślak-Malinowska, J. Lambek, M. Sadrzadeh, ...)
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- product pregroups
  (G. Kobele, T. Kusalik)

- tupled pregroups
  (C. Casadio, A. Kiślak-Malinowska, E. Stabler)
Let \((\mathbb{P}, \leq)\) be a (finite) poset. Elements of \(\mathbb{P}\) are called **atomic (basic) types.** Terms are of the form \(a^{(n)}\), for any \(a \in \mathbb{P}\) and \(n \in \mathbb{Z}\). A **type** is a finite sequence of terms.
Let \((P, \leq)\) be a (finite) poset. Elements of \(P\) are called atomic (basic) types. Terms are of the form \(a^{(n)}\), for any \(a \in P\) and \(n \in \mathbb{Z}\). A type is a finite sequence of terms.

In **TPG-expressions** we use tuples of typed strings. Usually the expressions of TPGs are written:

\[
\left( \begin{array}{ccc}
t_1 & \ldots & t_k \\
s_1 & \ldots & s_k \\
\end{array} \right)
\]
Let \((\mathbb{P}, \leq)\) be a (finite) poset. Elements of \(\mathbb{P}\) are called **atomic (basic) types**. **Terms** are of the form \(a^{(n)}\), for any \(a \in \mathbb{P}\) and \(n \in \mathbb{Z}\). **A type** is a finite sequence of terms.

In **TPG-expressions** we use tuples of typed strings. Usually the expressions of TPGs are written:

\[
\left( \begin{array}{c}
t_1 \\
\vdots \\
t_k \\
s_1 \\
\vdots \\
s_k 
\end{array} \right)
\]

\[
\left( \begin{array}{c}
w \\
\pi_3 \\
\text{who} \\
\epsilon \\
a \\
b \\
c 
\end{array} \right)
\left( \begin{array}{c}
A \\
B \\
C \\
a \\
b \\
c 
\end{array} \right)
\]
A merge operation applying to any pair of tuples is defined as follows:

\[
\begin{pmatrix}
t_1 & \ldots & t_i \\
s_1 & \ldots & s_i \\
\end{pmatrix} \cdot \begin{pmatrix}
t_{i+1} & \ldots & t_k \\
s_{i+1} & \ldots & s_k \\
\end{pmatrix} = \begin{pmatrix}
t_1 & \ldots & t_k \\
s_1 & \ldots & s_k \\
\end{pmatrix}
\]
A merge operation applying to any pair of tuples is defined as follows:

\[
\begin{pmatrix}
t_1 & \ldots & t_i \\
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\end{pmatrix}
\bullet
\begin{pmatrix}
t_{i+1} & \ldots & t_k \\
s_{i+1} & \ldots & s_k
\end{pmatrix}
= \begin{pmatrix}
t_1 & \ldots & t_k \\
s_1 & \ldots & s_k
\end{pmatrix}
\]

An operation of deleting i-th coordinate, for any k-tuple \( k > 0 \) and any \( 1 \leq i \leq k \) is defined as follows:

\[
\begin{pmatrix}
t_1 & \ldots & t_k \\
s_1 & \ldots & s_k
\end{pmatrix}
\_i
= \begin{pmatrix}
t_1 & \ldots & t_{i-1} & t_{i+1} & \ldots & t_k \\
s_1 & \ldots & s_{i-1} & s_{i+1} & \ldots & s_k
\end{pmatrix}
\]
A tupled pregroup grammar (TPG) is a quintuple $G = \langle \Sigma, \mathcal{P}, \leq, \mathbb{I}, s \rangle$, such that

- $\Sigma$ is a nonempty alphabet,
- $\mathcal{P}$ is a set of atomic (basic) types partially ordered by $\leq$,
- $\mathbb{I} \subset (\mathcal{T}_\mathcal{P} \times \Sigma^*)^*$,
- and $s \in \mathcal{P}$ is a designated type.
Let us define a binary relation on tupled pregroup expressions, denoted by \( \Rightarrow \) that holds in the following cases:
(for any tuples \( e_1, e_2 \) and sequence of tuples \( \alpha, \beta \))
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\[
(Mrg) \quad \alpha \ e_1 \ e_2 \ \beta \Rightarrow \ \alpha \ e_1 \bullet e_2 \ \beta
\]
Let us define a binary relation on tupled pregroup expressions, denoted by $\Rightarrow$ that holds in the following cases:
(for any tuples $e_1, e_2$ and sequence of tuples $\alpha, \beta$)

\[(Mrg)\quad \alpha \ e_1 \ e_2 \ \beta \Rightarrow \alpha \ e_1 \bullet e_2 \ \beta\]

\[(Move)\quad \alpha \left( \begin{array}{c} t_1 \cdots t_k \\ s_1 \cdots s_k \end{array} \right) \ \beta \Rightarrow \alpha \left( \begin{array}{c} t_i t_j \\ s_i s_j \end{array} \right) \bullet \left( \begin{array}{c} t_1 \cdots t_k \\ s_1 \cdots s_k \end{array} \right)_{-i-j} \ \beta\]
Tupled pregroup grammars - TPGs

\[(GCon)\quad \alpha \left( \ldots \ x a^{(n)} b^{(n+1)} y \atop s \ \ldots \right) \beta \Rightarrow \alpha \left( \ldots \ xy \atop s \ \ldots \right) \beta\]
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Tupled pregroup grammars - TPGs

\[(GCon)\quad \alpha \left( \ldots \ x^{a(n)}b^{(n+1)}y \atop s \right) \beta \Rightarrow \alpha \left( \ldots \ xy \atop s \right) \beta\]

\[(GExp)\quad \alpha \left( \ldots \ xy \atop s \right) \beta \Rightarrow \alpha \left( \ldots \ x^{a(n+1)}b^{(n)}y \atop s \right) \beta\]
The list of types

Types used in our examples:

- $s_1$: type of a sentence in a present tense
- $\pi_3$: third person subject
  - For example, *he*, *she*, *it*
- $\pi_3$: the trace of third person subject
- $o_4$: singular direct object
- $o_4$: the trace of singular direct object
The list of types

\( n \) type of a noun singular
for example *boy, author, book*

\( N \) type of a complete noun phrase (singular)
for example *a boy, the author, John, an interesting book*
We assume \( N \leq \pi_3 \) and \( N \leq o_4 \)

\( \bar{N} \) the trace of a complete noun phrase (singular)
We assume \( \bar{N} \leq \pi_3 \) and \( \bar{N} \leq o_4 \)

\( w \) type of relative pronoun
for example *who, which, that*

\( \bar{w} \) type of possessive relative phrase
for example *whose author, whose wise mother*
Dictionary

Let the dictionary (showing only tuples used in our examples) be as follows:

\[
I = \{ (Nn^\ell, \text{the}) (Nn^\ell, \text{a}) (n, \text{mother}) (n, \text{book}) (n, \text{boy}) \\
(N, \text{Mary}) (N, \text{John}) (\pi^r s_1 o_4^l, \text{likes}) (\pi^r s_1 o_4^l, \text{likes}) (\pi^r s_1 o_4^l, \text{likes}) \\
(\pi^r s_2 o_4^l, \text{liked}) (\pi^r s_2 o_4^l, \text{liked}) (\pi^r s_2 o_4^l, \text{liked}) (w, N, \epsilon) \\
w, o_4, \epsilon (\bar{w} n^\ell, N, \epsilon) (N^r N s^l w^l, \epsilon) (N^r N s^l \bar{w}^l, \epsilon) \ldots \}
\]
How to interpret tuples?

- $\left( \begin{array}{c} n \\ mother \end{array} \right)$ - this one means that the word *mother* has a type $n$ of a noun singular.
How to interpret tuples?

- \((n_{mother})\) - this one means that the word *mother* has a type \(n\) of a noun singular.

- \((Nn^\ell_{the})\) - the type of the determiner *the* is \(Nn^\ell\) and means that it will become a complete noun phrase singular when a noun singular is attached on its right (as for example *the mother*).
How to interpret tuples?

- \( \left( \begin{array}{c} n \\ mother \end{array} \right) \) - this one means that the word *mother* has a type \( n \) of a noun singular.

- \( \left( \begin{array}{c} Nn^l \\ the \end{array} \right) \) - the type of the determiner *the* is \( Nn^l \) and means that it will become a complete noun phrase singular when a noun singular is attached on its right (as for example *the mother*).

- \( \left( \begin{array}{c} \pi_3^r S_1 \omega_4^l \\ likes \end{array} \right) \) - this will become a sentence after attaching third person subject on its left and direct object on its right.
How to interpret tuples?

- \(( \begin{array}{c} w \\ whom \\ \epsilon \end{array} \begin{array}{c} o_4 \\ \ell \end{array} )\) - this one tells us that a relative pronoun whom put in the sentence will replace a direct object leaving the trace of a direct object in the context, as in *a boy whom John likes*.
How to interpret tuples?

- \( \left( \begin{array}{c} w \\ whom \\ \frac{o_4}{\epsilon} \end{array} \right) \) - this one tells us that a relative pronoun whom put in the sentence will replace a direct object leaving the trace of a direct object in the context, as in *a boy whom John likes _*.

- \( \left( \begin{array}{c} N^rNs^\ell w^\ell \\ \frac{\epsilon}{\epsilon} \end{array} \right) \) - this one is responsible for word order in the sentence with a relative pronoun.
Our example **John likes Mary** with the following lexical entries:

\[
\begin{pmatrix}
N \\
John
\end{pmatrix}
\begin{pmatrix}
N \\
Mary
\end{pmatrix}
\begin{pmatrix}
\pi^r_3 s^l_1 o^l_4 \\
likes
\end{pmatrix}
\]

Using the rules of \((Mrg)\), \((Move)\) and \((GCon)\) we can justify the correctness of this sentence with the following derivation:

\[
\begin{pmatrix}
N \\
John
\end{pmatrix}
\begin{pmatrix}
\pi^r_3 s^l_1 o^l_4 \\
likes
\end{pmatrix}
\begin{pmatrix}
N \\
Mary
\end{pmatrix}
\Rightarrow
\]

\((Mrg \text{ on the second and third tuples})\)
John likes Mary

Our example **John likes Mary** with the following lexical entries:

\[
\left( \begin{array}{c} N \\ John \end{array} \right) \left( \begin{array}{c} N \\ Mary \end{array} \right) \left( \begin{array}{c} \pi^r_3 s^l_1 o^l_4 \\ \text{likes} \end{array} \right)
\]

Using the rules of \((Mrg)\), \((Move)\) and \((GCon)\) we can justify the correctness of this sentence with the following derivation:

\[
\left( \begin{array}{c} N \\ John \end{array} \right) \left( \begin{array}{c} \pi^r_3 s^l_1 o^l_4 \\ \text{likes} \end{array} \right) \left( \begin{array}{c} N \\ Mary \end{array} \right) \Rightarrow \\
(Mrg \text{ on the second and third tuples})
\]

\[
\left( \begin{array}{c} N \\ John \end{array} \right) \left( \begin{array}{c} \pi^r_3 s^l_1 o^l_4 \\ \text{likes} \end{array} \right) \left( \begin{array}{c} N \\ Mary \end{array} \right) \Rightarrow \ N \leq o_4
\]

(making use of partial order)
John likes Mary

\[
\begin{pmatrix}
N \\
John
\end{pmatrix}
\begin{pmatrix}
\pi_3^r s_1 o_4^l \\
\text{likes} \\
\text{Mary}
\end{pmatrix}
\Rightarrow
\]

(Move on coordinates of the second tuple)
Example 1

John likes Mary

\[
\begin{pmatrix}
N \\
\text{John}
\end{pmatrix}
\begin{pmatrix}
\pi_3^r s_1 o_4^l \\
\text{likes}\end{pmatrix}
\begin{pmatrix}
o_4 \\
\text{Mary}
\end{pmatrix}
\Rightarrow
\]

(Move on coordinates of the second tuple)

\[
\begin{pmatrix}
N \\
\text{John}
\end{pmatrix}
\begin{pmatrix}
\pi_3^r s_1 o_4^l o_4 \\
\text{likes}\end{pmatrix}
\begin{pmatrix}
\text{Mary}
\end{pmatrix}
\Rightarrow
\]

(GCon in the second tuple)
John likes Mary

\[
\left( \begin{array}{c}
N \\
John
\end{array} \right) \left( \begin{array}{cc}
\pi^r_3 s_1 o_4^l \\
likes \\
Mary
\end{array} \right) \Rightarrow
\]

(Move on coordinates of the second tuple)

\[
\left( \begin{array}{c}
N \\
John
\end{array} \right) \left( \begin{array}{cc}
\pi^r_3 s_1 o_4^l o_4 \\
likes \\
Mary
\end{array} \right) \Rightarrow
\]

(GCon in the second tuple)

\[
\left( \begin{array}{c}
N \\
John
\end{array} \right) \left( \begin{array}{cc}
\pi^r_3 s_1 \\
likes \\
Mary
\end{array} \right) \Rightarrow
\]

(Mrg on the first and the second tuple)
John likes Mary

\[
\begin{pmatrix}
N & \pi_3^r s_1 \\
John & likes Mary
\end{pmatrix} \Rightarrow N \leq \pi_3
\]

(making use of partial order)
John likes Mary

\[
\left( \begin{array}{c}
N \\
\text{John likes Mary}
\end{array} \right) \Rightarrow \left( \begin{array}{c}
\pi_3^r S_1 \\
\text{John likes Mary}
\end{array} \right)
\]

(making use of partial order)

\[
\left( \begin{array}{c}
\pi_3 \\
\text{John likes Mary}
\end{array} \right) \Rightarrow \left( \begin{array}{c}
\pi_3 \\
\text{John likes Mary}
\end{array} \right)
\]

(Move on coordinates of the tuple)
Example 1

John likes Mary

\[
\begin{pmatrix}
N & \pi_3^r s_1 \\
John & likes Mary
\end{pmatrix}
\Rightarrow
N \leq \pi_3
\]
(making use of partial order)

\[
\begin{pmatrix}
\pi_3 & \pi_3^r s_1 \\
John & likes Mary
\end{pmatrix}
\Rightarrow
(Move on coordinates of the tuple)
\]

\[
\begin{pmatrix}
\pi_3 \pi_3^r s_1 \\
John likes Mary
\end{pmatrix}
\Rightarrow
(GCon within the tuple)
John likes Mary

\[
\left( \begin{array}{c} N \\ \pi_3^r s_1 \\ \text{John likes Mary} \end{array} \right) \Rightarrow N \leq \pi_3
\]

(making use of partial order)

\[
\left( \begin{array}{c} \pi_3 \\ \pi_3^r s_1 \\ \text{John likes Mary} \end{array} \right) \Rightarrow \text{(Move on coordinates of the tuple)}
\]

\[
\left( \begin{array}{c} \pi_3 \pi_3^r s_1 \\ \text{John likes Mary} \end{array} \right) \Rightarrow \text{(GCon within the tuple)}
\]

\[
\left( \begin{array}{c} s_1 \\ \text{John likes Mary} \end{array} \right)
\]
John likes Mary - adding semantics

\[\text{John likes Mary}\]

\[\pi_3^r s_1 o_4^l \Rightarrow s_1 \text{ (here } N \leq o_4 \text{ and } N \leq \pi_3)\]
John likes Mary - adding semantics

\[ \begin{array}{ccc}
\text{john} & x_1 & f & x_2 & \text{mary} \\
\text{John} & \text{likes} & \text{Mary} \\
N & \pi^r_3 s_1 o^l_4 & N & \Rightarrow s_1 \text{ (here } N \leq o_4 \text{ and } N \leq \pi_3 \text{)}
\end{array} \]

We add translation to semantic types in the following way:

\[ \begin{array}{ccc}
\text{John} & N & \text{john} \\
\text{Mary} & N & \text{mary} \\
\text{likes} & \pi^r_3 s_1 o^l_4 & f(x_1, x_2) = \text{like}(x_1, x_2)
\end{array} \]
John likes Mary - adding semantics

\[ john \quad x_1 \quad f \quad x_2 \quad mary \]

:\textit{John likes Mary} \\
\[ N \quad \pi_3^r \quad s_1 \quad o_4^l \quad N \quad \Rightarrow s_1 \ (\text{here } N \leq o_4 \text{ and } N \leq \pi_3) \]
John likes Mary - adding semantics

\[ john \ x_1 \ f \ x_2 \ mary \]
\[ John \ likes \ Mary \]
\[ N \ \pi_3 s_1 \ o_4^l \ N \Rightarrow s_1 \] (here \( N \leq o_4 \) and \( N \leq \pi_3 \))

Contractions according to underlinks define the substitutions. Thus in our examples we get:

\[ x_1 \rightarrow john \]
\[ x_2 \rightarrow mary \]
John likes Mary - adding semantics

\[\text{John} \quad \text{likes} \quad \text{Mary}\]

\[N \quad \pi_3^r s_1 o_4^l \quad N \quad \Rightarrow s_1 \quad (\text{here} \ N \leq o_4 \text{ and } N \leq \pi_3)\]

Contractions according to underlinks define the substitutions. Thus in our examples we get:

\[x_1 \rightarrow \text{John}\]
\[x_2 \rightarrow \text{Mary}\]

Now we are able to compute the semantics of the sentence \textbf{John likes Mary} by substitution in the following way:

\[f(x_1, x_2) = \text{like}(x_1, x_2) = \text{like}(\text{John}, \text{Mary})\]
the boy who likes Mary

Consider the string the boy who likes Mary with the following tuples:

\[
\begin{array}{l}
(Nn^\ell) \quad (n \quad \text{boy}) \quad (N \quad \text{Mary}) \quad (\frac{\pi^r s_1 o^\ell_4}{\text{likes}}) \quad (w \quad \text{who} \quad N \quad \epsilon) \\
(N^r Ns^\ell w^\ell \quad \epsilon)
\end{array}
\]

Then the derivation looks as follows:

\[
\begin{array}{l}
(Nn^\ell) \quad (n \quad \text{boy}) \quad (N^r Ns^\ell w^\ell \quad \epsilon) \quad (w \quad \text{who} \quad N \quad \epsilon) \quad (\frac{\pi^r s_1 o^\ell_4}{\text{likes}}) \\
(N \quad \text{Mary}) \quad \Rightarrow \quad N \leq o_4
\end{array}
\]
Consider the string **the boy who likes Mary** with the following tuples:

\[
\left( \begin{array}{c}
Nn^\ell \\
\text{the}
\end{array} \right) \quad \left( \begin{array}{c}
n^r \\
\text{boy}
\end{array} \right) \quad \left( \begin{array}{c}
N \\
\text{Mary}
\end{array} \right) \quad \left( \begin{array}{c}
\pi^r_3 s_1 o^\ell_4 \\
\text{likes}
\end{array} \right) \quad \left( \begin{array}{c}
w \\
\text{who}
\end{array} \right) \quad \left( \begin{array}{c}
N \\
\epsilon
\end{array} \right)
\]

Then the derivation looks as follows:

\[
\left( \begin{array}{c}
Nn^\ell \\
\text{the}
\end{array} \right) \quad \left( \begin{array}{c}
n^r \\
\text{boy}
\end{array} \right) \quad \left( \begin{array}{c}
N \\
\epsilon
\end{array} \right) \quad \left( \begin{array}{c}
w \\
\text{who}
\end{array} \right) \quad \left( \begin{array}{c}
\pi^r_3 s_1 o^\ell_4 \\
\text{likes}
\end{array} \right) \quad \Rightarrow \quad N \leq o_4
\]

\[
\left( \begin{array}{c}
Nn^\ell \\
\text{the}
\end{array} \right) \quad \left( \begin{array}{c}
n^r \\
\text{boy}
\end{array} \right) \quad \left( \begin{array}{c}
N \\
\epsilon
\end{array} \right) \quad \left( \begin{array}{c}
w \\
\text{who}
\end{array} \right) \quad \left( \begin{array}{c}
\pi^r_3 s_1 o^\ell_4 \\
\text{likes}
\end{array} \right) \quad \Rightarrow \quad o_4
\]
the boy who likes Mary

\[
\left( \begin{array}{c}
Nn^\ell n \\
\text{the boy}
\end{array} \right) \left( \begin{array}{c}
N^r Ns^\ell w^\ell \\
\epsilon
\end{array} \right) \left( \begin{array}{c}
w \\
\text{who}
\end{array} \right) \left( \begin{array}{c}
N \\
\epsilon
\end{array} \right) \left( \begin{array}{c}
\pi_3^r s_1 o_4^\ell o_4 \\
\text{likes Mary}
\end{array} \right) \Rightarrow
\]
the boy who likes Mary

\[
\left( \begin{array}{c}
Nn^\ell n \\
the \ boy
\end{array} \right) \left( \begin{array}{c}
N^rNs^\ell w^\ell \\
\epsilon
\end{array} \right) \left( \begin{array}{c}
w \\
who
\end{array} \right) \left( \frac{N}{\epsilon} \right) \left( \frac{\pi^r_3 s_1 o_4^\ell o_4}{\text{likes Mary}} \right) \Rightarrow
\]

\[
\left( \begin{array}{c}
N^\ell \\
the \ boy
\end{array} \right) \left( \begin{array}{c}
N^rNs^\ell w^\ell \\
\epsilon
\end{array} \right) \left( \begin{array}{c}
w \\
who
\end{array} \right) \left( \frac{N}{\epsilon} \right) \left( \frac{\pi^r_3 s_1}{\text{likes Mary}} \right) \Rightarrow
\]
the boy who likes Mary

\[
\left( N n^\ell n \right) \cdot \left( N^r N s^\ell w^\ell \right) \cdot \left( w \; \frac{N}{\epsilon} \right) \cdot \left( \frac{\pi^r s_1 o_4^\ell o_4^\ell}{\text{likes Mary}} \right) \Rightarrow
\]

\[
\left( N \right) \cdot \left( N^r N s^\ell w^\ell \right) \cdot \left( w \; \frac{N}{\epsilon} \right) \cdot \left( \frac{\pi^r s_1}{\text{likes Mary}} \right) \Rightarrow
\]

\[
\left( N \right) \cdot \left( N^r N s^\ell w^\ell \right) \cdot \left( w \; \frac{N}{\epsilon} \; \frac{\pi^r s_1}{\text{likes Mary}} \right) \Rightarrow \frac{N}{\pi_3} \leq \frac{\pi_3}{\pi_3}
\]
the boy who likes Mary

(\(Nn^\ell n\)) \((\ N^r Ns^\ell w^\ell\)) \((\ w \ N \ \epsilon\)) \((\ \frac{\pi^r s_1 o_4}{\ell} o_4 \likes Mary\)) \Rightarrow

(\(N\)) \((\ N^r Ns^\ell w^\ell\)) \((\ w \ N \ \epsilon\)) \((\ \frac{\pi^r s_1}{\ell} \likes Mary\)) \Rightarrow

(\(N\)) \((\ N^r Ns^\ell w^\ell\)) \((\ w \ N \ \epsilon\)) \((\ \frac{\pi^r s_1}{\ell} \likes Mary\)) \Rightarrow \(N \leq \pi_3\)

(\(NN^r Ns^\ell w^\ell\)) \((\ \frac{\pi_3 \pi^r s_1}{\ell} \likes Mary \ w \)) \Rightarrow
the boy who likes Mary

\[
\left( \begin{array}{c}
\text{the boy} \\
N_n^l \epsilon
\end{array} \right) \left( \begin{array}{c}
\text{who} \\
N_s^l \epsilon
\end{array} \right) \left( \begin{array}{c}
\text{likes Mary} \\
\frac{\pi_s^r \epsilon}{o_4^l}
\end{array} \right) \Rightarrow
\]

\[
\left( \begin{array}{c}
\text{the boy} \\
N \epsilon
\end{array} \right) \left( \begin{array}{c}
\text{who} \\
N_s^l \epsilon
\end{array} \right) \left( \begin{array}{c}
\text{likes Mary} \\
\frac{\pi_s^r \epsilon}{\pi_3^l}
\end{array} \right) \Rightarrow
\]

\[
\left( \begin{array}{c}
\text{the boy} \\
N \epsilon
\end{array} \right) \left( \begin{array}{c}
\text{who} \\
N_s^l \epsilon
\end{array} \right) \left( \begin{array}{c}
\text{likes Mary} \\
\frac{\pi_s^r \epsilon}{\pi_3^l}
\end{array} \right) \Rightarrow \frac{N}{\pi_3^l}
\]

\[
\left( \begin{array}{c}
\text{the boy} \\
NN^r \epsilon
\end{array} \right) \left( \begin{array}{c}
\text{likes Mary} \\
\frac{\pi_3^r \pi_s^l}{\pi_3^r}
\end{array} \right) \Rightarrow
\]

\[
\left( \begin{array}{c}
\text{the boy} \\
N_s^l \epsilon
\end{array} \right) \left( \begin{array}{c}
\text{likes Mary} \\
\frac{s_1 \epsilon}{\pi_3^r}
\end{array} \right) \Rightarrow
\]

\[
\left( \begin{array}{c}
\text{the boy} \\
N_s^l \epsilon
\end{array} \right) \left( \begin{array}{c}
\text{likes Mary} \\
\frac{s_1 \epsilon}{\pi_3^r}
\end{array} \right) \Rightarrow
\]

\[
\left( \begin{array}{c}
\text{the boy} \\
N_s^l \epsilon
\end{array} \right) \left( \begin{array}{c}
\text{likes Mary} \\
\frac{s_1 \epsilon}{\pi_3^r}
\end{array} \right) \Rightarrow
\]

\[
\left( \begin{array}{c}
\text{the boy} \\
N_s^l \epsilon
\end{array} \right) \left( \begin{array}{c}
\text{likes Mary} \\
\frac{s_1 \epsilon}{\pi_3^r}
\end{array} \right) \Rightarrow
\]

\[
\left( \begin{array}{c}
\text{the boy} \\
N_s^l \epsilon
\end{array} \right) \left( \begin{array}{c}
\text{likes Mary} \\
\frac{s_1 \epsilon}{\pi_3^r}
\end{array} \right) \Rightarrow
\]

\[
\left( \begin{array}{c}
\text{the boy} \\
N_s^l \epsilon
\end{array} \right) \left( \begin{array}{c}
\text{likes Mary} \\
\frac{s_1 \epsilon}{\pi_3^r}
\end{array} \right) \Rightarrow
\]

\[
\left( \begin{array}{c}
\text{the boy} \\
N_s^l \epsilon
\end{array} \right) \left( \begin{array}{c}
\text{likes Mary} \\
\frac{s_1 \epsilon}{\pi_3^r}
\end{array} \right) \Rightarrow
\]

\[
\left( \begin{array}{c}
\text{the boy} \\
N_s^l \epsilon
\end{array} \right) \left( \begin{array}{c}
\text{likes Mary} \\
\frac{s_1 \epsilon}{\pi_3^r}
\end{array} \right) \Rightarrow
\]

\[
\left( \begin{array}{c}
\text{the boy} \\
N_s^l \epsilon
\end{array} \right) \left( \begin{array}{c}
\text{likes Mary} \\
\frac{s_1 \epsilon}{\pi_3^r}
\end{array} \right) \Rightarrow
\]

\[
\left( \begin{array}{c}
\text{the boy} \\
N_s^l \epsilon
\end{array} \right) \left( \begin{array}{c}
\text{likes Mary} \\
\frac{s_1 \epsilon}{\pi_3^r}
\end{array} \right) \Rightarrow
\]

\[
\left( \begin{array}{c}
\text{the boy} \\
N_s^l \epsilon
\end{array} \right) \left( \begin{array}{c}
\text{likes Mary} \\
\frac{s_1 \epsilon}{\pi_3^r}
\end{array} \right) \Rightarrow
\]

\[
\left( \begin{array}{c}
\text{the boy} \\
N_s^l \epsilon
\end{array} \right) \left( \begin{array}{c}
\text{likes Mary} \\
\frac{s_1 \epsilon}{\pi_3^r}
\end{array} \right) \Rightarrow
\]

\[
\left( \begin{array}{c}
\text{the boy} \\
N_s^l \epsilon
\end{array} \right) \left( \begin{array}{c}
\text{likes Mary} \\
\frac{s_1 \epsilon}{\pi_3^r}
\end{array} \right) \Rightarrow
\]

\[
\left( \begin{array}{c}
\text{the boy} \\
N_s^l \epsilon
\end{array} \right) \left( \begin{array}{c}
\text{likes Mary} \\
\frac{s_1 \epsilon}{\pi_3^r}
\end{array} \right) \Rightarrow
\]

\[
\left( \begin{array}{c}
\text{the boy} \\
N_s^l \epsilon
\end{array} \right) \left( \begin{array}{c}
\text{likes Mary} \\
\frac{s_1 \epsilon}{\pi_3^r}
\end{array} \right) \Rightarrow
\]

\[
\left( \begin{array}{c}
\text{the boy} \\
N_s^l \epsilon
\end{array} \right) \left( \begin{array}{c}
\text{likes Mary} \\
\frac{s_1 \epsilon}{\pi_3^r}
\end{array} \right) \Rightarrow
\]

\[
\left( \begin{array}{c}
\text{the boy} \\
N_s^l \epsilon
\end{array} \right) \left( \begin{array}{c}
\text{likes Mary} \\
\frac{s_1 \epsilon}{\pi_3^r}
\end{array} \right) \Rightarrow
\]

\[
\left( \begin{array}{c}
\text{the boy} \\
N_s^l \epsilon
\end{array} \right) \left( \begin{array}{c}
\text{likes Mary} \\
\frac{s_1 \epsilon}{\pi_3^r}
\end{array} \right) \Rightarrow
\]

\[
\left( \begin{array}{c}
\text{the boy} \\
N_s^l \epsilon
\end{array} \right) \left( \begin{array}{c}
\text{likes Mary} \\
\frac{s_1 \epsilon}{\pi_3^r}
\end{array} \right) \Rightarrow
\]
the boy who likes Mary

\[
\begin{pmatrix}
Ns^\ell w^\ell & s_1 & w \\
the boy & likes Mary & who
\end{pmatrix}
\Rightarrow
\]
the boy who likes Mary

\[
\begin{pmatrix}
Ns^l w^l & s_1 & w \\
the \ boy & likes \ Mary & who
\end{pmatrix} \Rightarrow
\]

\[
\begin{pmatrix}
Ns^l w^l w & s_1 \\
the \ boy \ who & likes \ Mary
\end{pmatrix} \Rightarrow
\]
the boy who likes Mary

\[
\begin{pmatrix}
Ns^l w^l & s_1 & w \\
\text{the boy} & \text{likes Mary} & \text{who}
\end{pmatrix} \Rightarrow
\]

\[
\begin{pmatrix}
Ns^l w^l w & s_1 \\
\text{the boy who} & \text{likes Mary}
\end{pmatrix} \Rightarrow
\]

\[
\begin{pmatrix}
Ns^l & s_1 \\
\text{the boy who} & \text{likes Mary}
\end{pmatrix} \Rightarrow
\]
the boy who likes Mary

\[
\left( \begin{array}{c} Ns^l w^l \\ \text{the boy} \\ \text{likes Mary} \\ \text{who} \end{array} \right) \Rightarrow \\
\left( \begin{array}{c} Ns^l \ell w^l w \\ \text{the boy} \\ \text{who} \\ \text{likes Mary} \end{array} \right) \Rightarrow \\
\left( \begin{array}{c} Ns^l \\ \text{the boy} \\ \text{who} \\ \text{likes Mary} \end{array} \right) \Rightarrow \\
\left( \begin{array}{c} Ns^l s_1 \\ \text{the boy who likes Mary} \end{array} \right) \Rightarrow s_1 \leq s
\]
the boy who likes Mary

\[
\left( Ns^l w^l \begin{array}{c} s_1 \\
\text{the boy likes Mary} \\
\text{who} \end{array} w \right) \Rightarrow
\]

\[
\left( Ns^l w^l w \begin{array}{c} s_1 \\
\text{the boy who} \\
\text{likes Mary} \end{array} \right) \Rightarrow
\]

\[
\left( Ns^l \begin{array}{c} s_1 \\
\text{the boy who} \\
\text{likes Mary} \end{array} \right) \Rightarrow
\]

\[
\left( Ns^l s_1 \begin{array}{c} \text{the boy who likes Mary} \end{array} \right) \Rightarrow s_1 \leq s
\]

\[
\left( N \begin{array}{c} \text{the boy who likes Mary} \end{array} \right)
\]

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the boy who likes Mary - adding semantics

\[ f \times \text{boy} \ y_1 \ h \ y_2 \ y_3 \ \epsilon \ \text{who} \ \epsilon \ \text{likes} \ \text{Mary} \]

Semantic types added are as follows:

<table>
<thead>
<tr>
<th>the</th>
<th>(Nn^l)</th>
<th>(f(x) = id(x) = x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>boy</td>
<td>(n)</td>
<td>boy</td>
</tr>
<tr>
<td>Mary</td>
<td>(N)</td>
<td>mary</td>
</tr>
<tr>
<td>likes</td>
<td>(\pi_3^r s_1 o_4^l)</td>
<td>(g(x_1, x_2) = \text{like}(x_1, x_2))</td>
</tr>
<tr>
<td>who</td>
<td>(w)</td>
<td>-</td>
</tr>
<tr>
<td>(\epsilon)</td>
<td>(N)</td>
<td>-</td>
</tr>
<tr>
<td>(\epsilon)</td>
<td>(N^r Ns^l w^l)</td>
<td>(h(y_1, y_2, y_3) = y_1 \land y_2)</td>
</tr>
</tbody>
</table>
the boy who likes Mary - adding semantics

\[
\begin{align*}
  f \ x & \quad \text{boy} & \quad y_1 & \quad h \ y_2 & \quad y_3 & - & - & \quad x_1 \ g \ x_2 & \quad \text{mary} \\
  \text{the} & \quad \text{boy} & \quad \epsilon & \quad \text{who} & \quad \epsilon & \quad \text{likes} & \quad \text{Mary} \\
  N \ n^l & \quad \text{N} & \quad N^r & \quad N \ s^l & \quad w^l & \quad w & \quad N & \quad \pi_3^r \ s_1 \ o_4^l & \quad N & \quad \Rightarrow & \quad N
\end{align*}
\]

Contractions according to the links define substitution:

\[
\begin{align*}
  x & \rightarrow \text{boy} \\
  x_1 & \rightarrow - \\
  x_2 & \rightarrow \text{mary} \\
  y_1 & \rightarrow f(x) \\
  y_2 & \rightarrow g(x_1, x_2)
\end{align*}
\]
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the boy who likes Mary - adding semantics

\[
f(x) = id(x) = x \\
\text{boy} \quad n \\
\text{Mary} \quad N \\
\text{likes} \quad \pi_3 s_1 o_4 \\
\text{who} \quad w \\
\epsilon \quad N \\
\epsilon \quad N^r N s^l w^l \\
h(y_1, y_2, y_3) = y_1 \land y_2
\]
the boy who likes Mary - adding semantics

\[
f(x) = \text{id}(x) = x
\]

\[
\text{boy} \quad \text{boy} \\
\text{Mary} \quad \text{mary} \\
\text{likes} \quad g(x_1, x_2) = \text{like}(x_1, x_2) \\
\text{who} \quad - \\
\epsilon \quad - \\
\epsilon \quad -
\]

\[
h(y_1, y_2, y_3) = y_1 \land y_2
\]
the boy who likes Mary - adding semantics

\[ f(x) = id(x) = x \]

\[ g(x_1, x_2) = \text{like}(x_1, x_2) \]

\[ h(y_1, y_2, y_3) = y_1 \land y_2 \]

\[ h(y_1, y_2, y_3) = y_1 \land y_2 = \]
the boy who likes Mary - adding semantics

\[
f(x) = \text{id}(x) = x
\]

\[
\text{boy} \quad n \quad \text{boy}
\]

\[
\text{Mary} \quad N \quad \text{mary}
\]

\[
\text{likes} \quad \pi_3 s_1 o^l_4 \quad g(x_1, x_2) = \text{like}(x_1, x_2)
\]

\[
\text{who} \quad w \quad -
\]

\[
\epsilon \quad N \quad -
\]

\[
\epsilon \quad N^r N s^l w^l \quad h(y_1, y_2, y_3) = y_1 \land y_2
\]

\[
h(y_1, y_2, y_3) = y_1 \land y_2 = f(x) \land g(x_1, x_2) =
\]
the boy who likes Mary - adding semantics

\[ f(x) = \text{id}(x) = x \]

\[ g(x_1, x_2) = \text{like}(x_1, x_2) \]

\[ h(y_1, y_2, y_3) = y_1 \land y_2 = f(x) \land g(x_1, x_2) = \text{id}(x) \land \text{like}(x_1, x_2) = \]
the boy who likes Mary - adding semantics

\[ f(x) = \text{id}(x) = x \]

\[ h(y_1, y_2, y_3) = y_1 \land y_2 = f(x) \land g(x_1, x_2) = \text{id}(x) \land \text{like}(x_1, x_2) = \text{id}(\text{boy}) \land \text{like}(\text{--}, \text{mary}) = \]
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the boy who likes Mary - adding semantics

\[ f(x) \cdot \text{boy} \cdot y_1 \cdot h \cdot y_2 \cdot y_3 = x_1 \cdot g(x_2) \cdot \text{mary} \]

\[ \begin{align*}
\text{the} &\quad Nn^l
\text{boy} &\quad n
\text{Mary} &\quad N
\text{likes} &\quad \pi_3^r s_1^l o_4^l
\text{who} &\quad w
\epsilon &\quad N
\epsilon &\quad N^r Ns^l w^l
\end{align*} \]

\[ \Rightarrow N \]

\[ h(y_1, y_2, y_3) = y_1 \land y_2 = f(x) \land g(x_1, x_2) = id(x) \land \text{like}(x_1, x_2) = id(\text{boy}) \land \text{like}(\_ , \text{mary}) = \text{boy} \land \text{like}(\_ , \text{mary}) \]
the boy whom Mary likes

\[
f(x) \quad \text{boy} \quad y_1 \quad h \quad y_2 \quad y_3 \quad \text{—} \quad \text{mary} \quad x_1 \quad g \quad x_2 \quad \text{—}
\]

\[
\text{the boy} \quad \epsilon \quad \text{whom Mary likes} \quad \epsilon
\]

\[
N \ n^l \quad n \quad N^r \ N \ s^l \ w^l \quad w \quad N \quad \pi^r_3 \ s_1 \ o^l_4 \quad o^l_4 \quad \Rightarrow \ N
\]

\[
\text{the} \quad N^n^l \quad f(x) = \text{id}(x) = x
\]

\[
\text{boy} \quad n \quad \text{boy}
\]

\[
\text{Mary} \quad N \quad \text{mary}
\]

\[
\text{likes} \quad \pi^r_3 \ s_1 \ o^l_4 \quad g(x_1, x_2) = \text{like}(x_1, x_2)
\]

\[
\text{whom} \quad w \quad \text{—}
\]

\[
\epsilon \quad o^l_4 \quad \text{—}
\]

\[
\epsilon \quad N^r \ N s^l \ w^l \quad h(y_1, y_2, y_3) = y_1 \land y_2
\]
the boy whom Mary likes

\[
f(x) \quad \text{boy} \quad y_1 \quad h \quad y_2 \quad y_3 \quad - \quad \text{mary} \quad x_1 g x_2 \quad -
\]

\[
\text{the boy} \quad \epsilon \quad \text{whom} \quad \text{Mary} \quad \text{likes} \quad \epsilon
\]

\[
N \quad n^\ell \quad n \quad N^r \quad N \quad s^\ell \quad w^\ell \quad w \quad N \quad \pi^r_3 \quad s_1 \quad o^l_4 \quad o_4 \quad \Rightarrow \quad N
\]

\[
\text{the} \quad N n^\ell \quad f(x) = \text{id}(x) = x
\]

\[
\text{boy} \quad n \quad \text{boy}
\]

\[
\text{Mary} \quad N \quad \text{mary}
\]

\[
\text{likes} \quad \pi^r_3 s_1 o^l_4 \quad g(x_1, x_2) = \text{like}(x_1, x_2)
\]

\[
\text{whom} \quad w \quad -
\]

\[
\epsilon \quad o_4 \quad -
\]

\[
\epsilon \quad N^r N s^\ell w^\ell \quad h(y_1, y_2, y_3) = y_1 \land y_2
\]

\[
h(y_1, y_2, y_3) = y_1 \land y_2 = f(x) \land g(x_1, x_2) = \text{id}(x) \land \text{like}(x_1, x_2) = \text{id(boy)} \land \text{like( mary, -)} = \text{boy} \land \text{like( mary, -)}
\]
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the boy whose mother liked John

\[
f(x) = id(x) \quad g(x_1, x_2) = \text{like}(x_1, x_2) \\
\text{have}(\neg, y_3 \land y_2) \quad N \Rightarrow
\]
the boy whose mother liked John

\[
\begin{align*}
\text{the boy} & \quad \epsilon \\
\text{whose mother} & \quad \epsilon \\
\text{liked John} & \quad \text{id(b o y)} \land \text{have(−, mother \land like(−, john))}
\end{align*}
\]

\[
\begin{align*}
h(y_1, y_2, y_3) &= y_1 \land \text{have}(−, y_3 \land y_2) = f(x) \land \text{have}(−, j(z) \land g(x_1, x_2)) = \\
&\quad \text{id}(x) \land \text{have}(−, \text{id}(z) \land \text{like}(x_1, x_2)) = \\
&\quad \text{id}(\text{boy}) \land \text{have}(−, \text{id}(\text{mother}) \land \text{like}(−, \text{john})) = \\
&\quad \text{boy} \land \text{have}(−, \text{mother} \land \text{like}(−, \text{john}))
\end{align*}
\]
the boy whose mother John liked

\[ f(x) = id(x) \]
\[ g(x_1, x_2) = \text{like}(x_1, x_2) \]
\[ j(z) = id(z) \]
\[ h(y_1, y_2, y_3) = y_1 \land \text{have}(\neg, y_3 \land y_2) \]
the boy whose mother John liked

\[
\begin{align*}
\text{the} & \quad Nn^l \quad f(x) = id(x) \\
\text{liked} & \quad \pi_3^r s_2 o_4^l \quad g(x_1, x_2) = \text{like}(x_1, x_2) \\
\text{whose} & \quad \text{wn}^l \quad j(z) = id(z) \\
\epsilon & \quad N \quad - \\
\epsilon & \quad N^r Ns^l \tilde{w}^l \quad h(y_1, y_2, y_3) = y_1 \land \text{have}(\neg, y_3 \land y_2) \\
\end{align*}
\]

\[
\begin{align*}
h(y_1, y_2, y_3) &= y_1 \land \text{have}(\neg, y_3 \land y_2) \equiv f(x) \land \text{have}(\neg, j(z) \land g(x_1, x_2)) = \\
&= id(x) \land \text{have}(\neg, id(z) \land \text{like}(x_1, x_2)) = \\
&= id(\text{boy}) \land \text{have}(\neg, id(\text{mother}) \land \text{like}(\text{john}, \neg)) = \\
&= \text{boy} \land \text{have}(\neg, \text{mother} \land \text{like}(\text{john}, \neg))
\end{align*}
\]
the boy who has a mother who liked John

\[ f \times \text{boy} \ y_1 \ h \ y_2 \ y_3 \quad \epsilon \quad \text{who} \ \epsilon \quad \text{has} \ \epsilon \quad x_1 \ g \ x_2 \ f \ y \ \text{mother} \ y_4 \ h \ y_5 \ y_6 \quad \epsilon \quad \text{who} \ \epsilon \quad \text{liked} \ \epsilon \quad \text{john} \]

\[ (y_1, y_2, y_3) = y_1 \land y_2 \]

\[ g(x_1, x_2) = \text{have}(x_1, x_2) \]

\[ h(y_4, y_5, y_6) = y_4 \land y_5 \]

\[ g(x_1, x_2) = \text{like}(x_1, x_2) \]
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\[ h(y_1, y_2, y_3) = y_1 \land y_2 \]

\[ id(x) \land \text{have}(x_1, x_2) = \text{id(boy)} \land \text{have}(\neg, h(y_4, y_5, y_6)) = \text{boy} \land \text{have}(\neg, y_4 \land y_5) = \text{boy} \land \text{have}(\neg, f(y) \land j(z_1, z_2)) = \text{boy} \land \text{have}(\neg, \text{id(y)} \land \text{like}(z_1, z_2)) = \text{boy} \land \text{have}(\neg, \text{id(mother)} \land \text{like}(\neg, \text{john})) = \text{boy} \land \text{have}(\neg, \text{mother} \land \text{like}(\neg, \text{john})) \]
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\( f(x) \)  |  \( \text{boy} \)  |  \( y_1 \)  |  \( h \)  |  \( y_2 \)  |  \( y_3 \)  |  \( \epsilon \)  |  \( \text{who} \)  |  \( \epsilon \)  |  \( x_1 \)  |  \( g \)  |  \( x_2 \)  |  \( f \)  |  \( y \)  |  \( \text{mother} \)  |  \( y_4 \)  |  \( h \)  |  \( y_5 \)  |  \( y_6 \)  |  \( \epsilon \)  |  \( \text{whom} \)  |  \( \text{John} \)  |  \( z_1 \)  |  \( j \)  |  \( z_2 \)  |  \( \epsilon \)  \\
\( N^n \)  |  \( n \)  |  \( N^r \)  |  \( N^s \)  |  \( w \)  |  \( w \)  |  \( N \)  |  \( \pi_3 s_1 o_4 \)  |  \( N^n \)  |  \( n \)  |  \( N^r \)  |  \( N^s \)  |  \( w \)  |  \( w \)  |  \( N \)  |  \( \pi_3 s_2 o_4 \)  |  \( N \)  

\[
\epsilon \quad N^r N^s w^l \quad \Rightarrow
\]

\[
\epsilon \quad N^r N^s w^l \quad h(y_1, y_2, y_3) = y_1 \land y_2
\]

\[
\text{has} \quad \pi_3 s_2 o_4 \quad g(x_1, x_2) = \text{have}(x_1, x_2)
\]

\[
\epsilon \quad N^r N^s w^l \quad h(y_4, y_5, y_6) = y_4 \land y_5
\]

\[
\text{liked} \quad \pi_3 s_2 o_4 \quad g(x_1, x_2) = \text{like}(x_1, x_2)
\]
the boy who has a mother whom John liked

\[
\begin{align*}
&f \circ \text{boy} = y_1 \land y_2 \\
&\text{has} = \pi_3^r s_2 o_4^l \\
&\epsilon \circ N^r N s^l w^l = h(y_1, y_2, y_3) = y_1 \land y_2 \\
&\epsilon \circ N^r N s^l w^l = g(x_1, x_2) = \text{have}(x_1, x_2) \\
&\epsilon \circ N^r N s^l w^l = h(y_4, y_5, y_6) = y_4 \land y_5 \\
&\epsilon \circ N^r N s^l w^l = g(x_1, x_2) = \text{like}(x_1, x_2)
\end{align*}
\]

\[
\begin{align*}
&h(y_1, y_2, y_3) = y_1 \land y_2 = f(x) \land g(x_1, x_2) = \\
&id(x) \land \text{have}(x_1, x_2) = \\
&id(\text{boy}) \land \text{have}(\epsilon, h(y_4, y_5, y_6)) = \\
\text{boy} \land \text{have}(\epsilon, y_4 \land y_5) = \\
\text{boy} \land \text{have}(\epsilon, f(y) \land j(z_1, z_2)) = \\
\text{boy} \land \text{have}(\epsilon, id(y) \land \text{like}(z_1, z_2)) = \\
\text{boy} \land \text{have}(\epsilon, id(\text{mother}) \land \text{like}(\text{john}, \epsilon)) = \\
\text{boy} \land \text{have}(\epsilon, \text{mother} \land \text{like}(\text{john}, \epsilon))
\end{align*}
\]
want(audience, finish(speaker, talk))
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Thank you!