PTAS for Minimax Approval Voting

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Definitions

Minimax Approval Voting (MAV)

- find a committee of fixed size k
- minimize the maximal Hamming distance from a vote

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Theorem. Minimax Approval Voting is NP-hard.

Proof: R.LeGrand. Analysis of the minimax procedure (2004).

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our result for Minimax Approval Voting

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Notation and definitions

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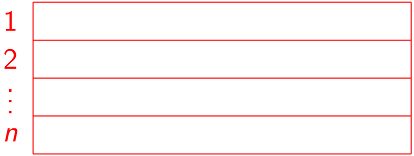
n – number of voters



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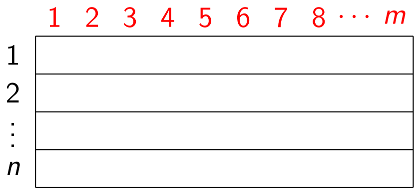

n – number of voters

m – number of candidates

Notation and definitions

	1	2	3	4	5	6	7	8	$\cdots$	m
1										
2										
$\vdots$	1	1	0	1	0	0	1	0	$\cdots$	0
n										

$= s_i$

n – number of voters

m – number of candidates

$s_i \in \{0, 1\}^m$ – a vote of voter i-th

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	1	2	3	4	5	6	7	8	$\cdots$	m
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2	1	0	0	1	0	1	0	0	$\cdots$	1
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n	0	1	0	1	1	1	0	0	⋯	0

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$d(x,y) = |\{j : x[j] \neq y[j]\}|$

$x, y \in \{0,1\}^m$

Notation and definitions

	1	2	3	4	5	6	7	8	$\cdots$	m
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Definicja 1. Closest String

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$$OPT = \min_{x \in \{0,1\}^m} \max_{i \in \{1,2,\ldots,n\}} d(x, s_i)$$

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Definicja 1. Closest String → Minimax Approval Voting

$$OPT = \min_{\substack{x \in \{0,1\}^m \\ x^{(1)}=k}} \max_{i \in \{1,2,\ldots,n\}} d(x, s_i)$$

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	1	2	3	4	5	6	7	8	$\cdots$	m
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Definicja 1. Minimax Approval Voting

$$OPT = \min_{\substack{x \in \{0,1\}^m \\ x^{(1)} = k}} \max_{i \in \{1,2,\ldots,n\}} d(x, s_i) = d(s_{OPT}, s_i)$$

This talk

Useful functions

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For all $Y \subseteq S, Y \neq \emptyset$

$$\begin{array}{|cccccccccc|}\hline 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & \cdots & 1 \\ \hline 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & \cdots & 1 \\ \hline 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & \cdots & 1 \\ \hline 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & \cdots & 1 \\ \hline \end{array} = Y$$

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1	1	1	0	1	1	0	0	$\cdots$	1	
1	0	0	1	0	1	0	0	$\cdots$	1	
1	0	0	1	0	1	0	0	$\cdots$	1	$= Y$
0	0	1	1	1	1	0	0	$\cdots$	1	
$*$	$*$	$*$	$*$	$*$	1	0	0	$\cdots$	1	$p(Y)$

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$$(p(Y))[j] = \begin{cases} 0 & \text{if } \forall_{y \in Y} \quad y[j] = 0 \\ 1 & \text{if } \forall_{y \in Y} \quad y[j] = 1 \\ * & \text{otherwise} \end{cases}$$

1	1	1	0	1	1	0	0	⋯	1
1	0	0	1	0	1	0	0	⋯	1
1	0	0	1	0	1	0	0	⋯	1
0	0	1	1	1	1	0	0	⋯	1

$= Y$

*	*	*	*	*	1	0	0	⋯	1

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0	0	1	1	1	1	0	0	$\cdots$	1	
*	*	*	*	*	1	0	0	$\cdots$	1	$p(Y)$
1	0	0	1	1	0	1	0	$\cdots$	1	s_{OPT}

$$ina(Y) = d(p''(Y), s''_{OPT})$$

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Lemat 2. ($ina(Y)$ is supermodular)

If we artificially extend the $ina(\cdot)$ function for the empty set: $ina(\emptyset) = 2 \cdot OPT$, then the $ina(\cdot)$ function is supermodular, i.e.,

$$\forall_{Y \subseteq Z \subseteq S} \quad \forall_{s \in S} \quad ina(Z) - ina(Z \cup \{s\}) \leqslant ina(Y) - ina(Y \cup \{s\})$$

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For any fixed $R \in \mathbb{N}_{\geqslant 1}$ there exists a subset $X \subseteq S, |X| = R$ such that

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We say such X is $\frac{OPT}{R}$-stable.

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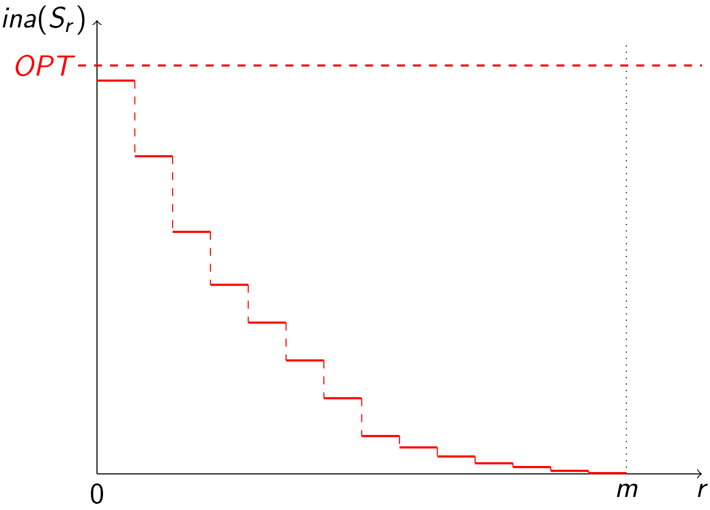

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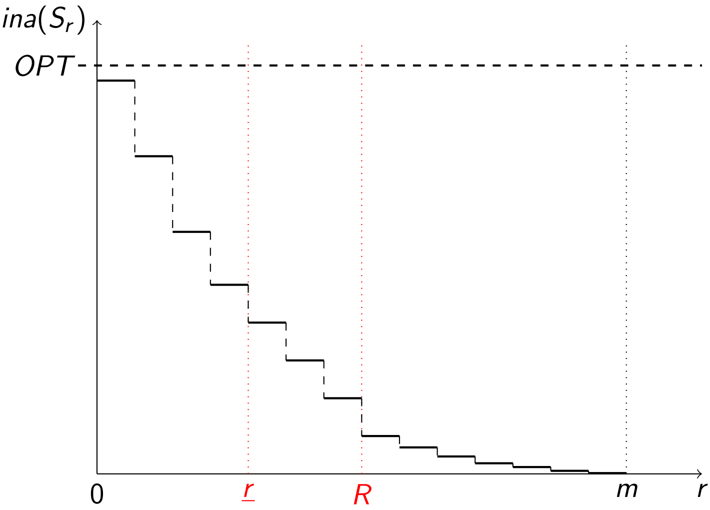

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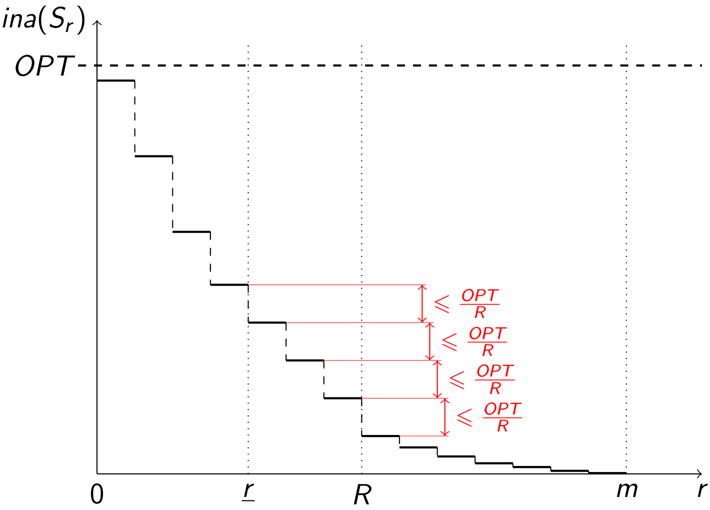

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for X from Lemma 3 we have

$$p^{(*)}(X) \leqslant |X| \cdot OPT = R \cdot OPT$$

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$x' = x[1] \cdot x[2] \cdot \ldots \cdot x[\beta]$
$x'' = x[\beta + 1] \cdot x[\beta + 2] \cdot \ldots \cdot x[m]$
$x = x' \cdot x''$

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If $X \subseteq S$ is $(\epsilon_1 \cdot OPT)$-stable, $k'' = (s''_{OPT})^{(1)}$ and z'' is a k''-completion of $(p(X))''$, then

$$\forall_{i \in \{1,2,\cdots,n\}} \quad d(s'_{OPT} \cdot z'', s_i) \leqslant$$

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$$\forall_{i \in \{1,2,\cdots,n\}} \quad d(s'_{OPT} \cdot z'', s_i) \leqslant (1 + 2\epsilon_1) \cdot OPT$$

Proof of Lemma 5.

	combinations											
index of a combination	1	2	3	4	5	6	7	8	9	10	11	12

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$(p(X))''[j]$	0	0	0	0	0	0	0	0	1	1	1	1
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$s_i''[j]$	0	1	0	1	0	1	0	1	0	1	0	1
$s_{OPT}''[j]$	0	0	1	1	0	0	1	1	0	0	1	1

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$s_{OPT}''[j]$	0	0	1	1	0	0	1	1	0	0	1	1
number of occurrences	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9	α_{10}	α_{11}	α_{12}

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$s_i''[j]$	0	1	0	1	0	1	0	1	0	1	0	1
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number of occurrences	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9	α_{10}	α_{11}	α_{12}
$d(z''[j], s_i''[j])$	0	1	0	1	1	0	1	0	1	0	1	0

Proof of Lemma 5.

	combinations											
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$\sum_{k=5}^{12} \alpha_k = \alpha_3 + \alpha_4 + \alpha_7 + \alpha_8 + \alpha_{11} + \alpha_{12}$ (number of 1's in z'' and s_{OPT}'' is equal)

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$d(s_{OPT}' \cdot z'', s_i) \leqslant (1 + 2\epsilon_1) \cdot OPT$ ■

An auxiliary optimization problem

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Definition. $IP(Y, k')$

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$$\min q \tag{1}$$

$$q \geqslant 0 \tag{4}$$

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$$(s')^{(1)} = k' \tag{2}$$

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Good approximation of star-part

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Good approximation of star-part

Lemma 6. (we can approximate s'_{OPT} "very well")

$\forall_{R \in \mathbb{N}_{\geqslant 1}, Y \subseteq S, |Y| \leqslant R, k' \in \mathbb{N}, \epsilon_2 > 0}$ we can find $(1 + 2\epsilon_2)$-approximation solution of $IP(Y, k')$ by solving the LP and considering polynomial many cases.

Sketch of proof

Case 1:

Case 2:

Case 3:

Sketch of proof

Case 1: $\beta \leqslant \frac{3R\ln(3n)}{(\epsilon_2)^2}$

There is less than $(3n)^{\frac{3R\ln(2)}{(\epsilon_2)^2}} \in Poly(n)$ cases.

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- use LP relaxation for IP
- randomized rounding
- k'-completion

PTAS for Minimax Approval Voting

1. **input:** $S \in (\{0,1\}^m)^n, 0 \leqslant k \leqslant m, R \in \mathbb{N}_{\geqslant 1}$
2. **output:** $s_{ALG} \in \{0,1\}^m$
3.
4.
5.
6.
7.
8.
9.
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for each R-element subset $Y \subseteq S$
    for each division k into two parts $k = k' + k''$
        $s''_{ALG} \leftarrow k''$-completion of $(p(Y))''$
    endfor
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PTAS for Minimax Approval Voting

```
input: S ∈ ({0,1}^m)^n, 0 ≤ k ≤ m, R ∈ ℕ_{≥1}
output: s_ALG ∈ {0,1}^m
for each R-element subset Y ⊆ S
    for each division k into two parts k = k' + k''
        s''_ALG ← k''-completion of (p(Y))''
        s'_ALG ← approximation solution of IP(Y, k')

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6. $s'_{ALG} \leftarrow$ approximation solution of $IP(Y, k')$
7. evaluate s_{ALG} by computing $\max_{i \in \{1,2,\ldots,n\}} d(s_i, s'_{ALG} \cdot s''_{ALG})$
8. **endfor**
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Theorem

$\forall_{\epsilon \in (0,1)}$ *we may compute a* $(1+\epsilon)$*-approximate solution to Minimax Approval Voting in* $O(Poly(n,m))$ *time.*

Questions?