# Harmonious Coloring of Hypergraphs

Bartłomiej Bosek
Theoretical Computer Science
Faculty of Mathematics and Computer Science
Jagiellonian University, Kraków, Poland
bosek@tcs.uj.edu.pl

Sebastian Czerwiński
Faculty of Mathematics, Computer Science,
and Econometrics
University of Zielona Góra, Zielona Góra, Poland
s.czerwinski@wmie.uz.zgora.pl

Jarosław Grytczuk
Theoretical Computer Science
Faculty of Mathematics and Computer Science
Jagiellonian University, Kraków, Poland
grytczuk@tcs.uj.edu.pl

Paweł Rzążewski
Faculty of Mathematics and Information Science
Warsaw University of Technology, Warszawa, Poland
p.rzazewski@mini.pw.edu.pl

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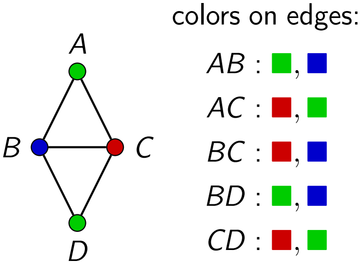
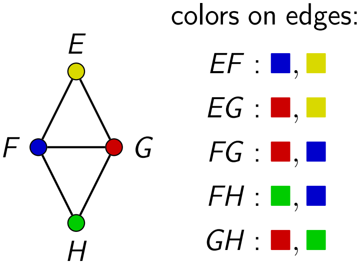
Warszawa, 30-31 January 2015

# Harmonious Coloring of Graphs

A harmonious coloring of a simple graph $G$ is a vertex coloring such that each pair of colors appears together on at most one edge.



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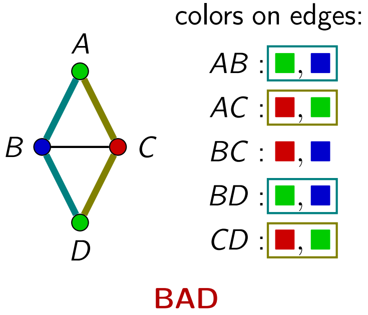


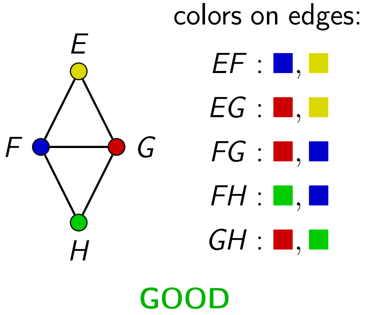

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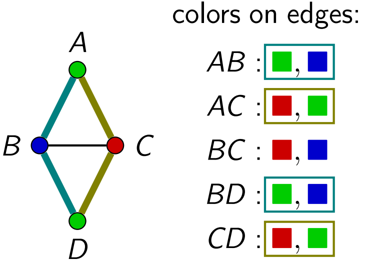
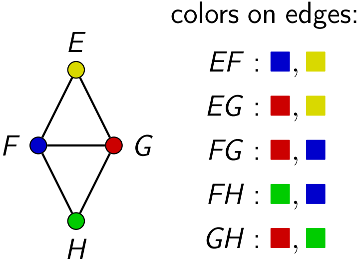


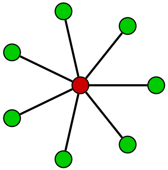

**BAD**




**GOOD**

The harmonious chromatic number $h(G)$ is the least number of colors in a harmonious coloring of a simple graph $G$.

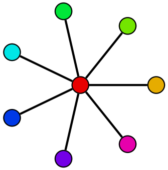
# Harmonious Coloring of Graphs



# Harmonious Coloring of Graphs

- the number $h(G)$ of colors necessary to harmoniously coloring of graph $G$ is not smaller than maximal degree of $G$ plus 1, i.e.,

$$h(G) \geqslant \Delta(G) + 1$$



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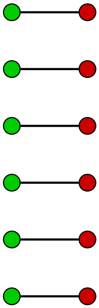

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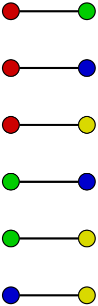
- the number $m$ of edges of graph $G$ must be not greater than the number of two-element subsets of set of colors, i.e., $\binom{h(G)}{2} \geqslant m$ and in consequence

$$h(G) \geqslant \sqrt{2m}$$



# Harmonious Coloring of Graphs

$\delta$-degenerated graph – each subgraph contains a vertex with degree at most $\delta$

**Theorem (Edwards & McDiarmid, 94')**

*A $\delta$-degenerated graph with $m$ edges and a maximal degree $\Delta$ is harmoniously colored by $2\sqrt{2\delta m} + 2(\delta - 1)\Delta$ colors.*

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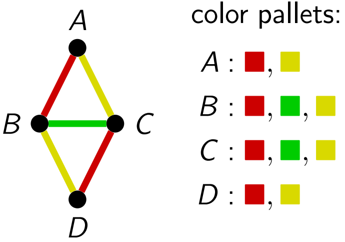
Planar graphs are 5-degenerated.

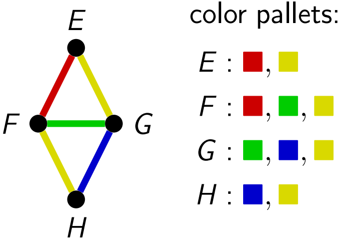
**Corollary**

A planar graph with $m$ edges and a maximal degree $\Delta$ is harmoniously colored by $\sqrt{40m} + 8\Delta$ colors.

# Vertex-distinguishing Edge Coloring

color palette occurring on vertex $v$ – a set of colors assigned to edges incidents to $v$

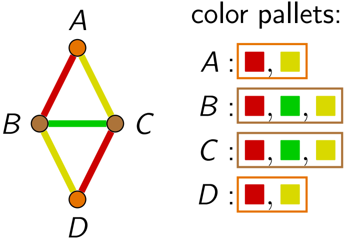


# Vertex-distinguishing Edge Coloring

color palette occurring on vertex $v$ – a set of colors assigned to edges incidents to $v$

vertex-distinguishing edge coloring – an edge coloring distinguishing color palettes occurring on different vertices

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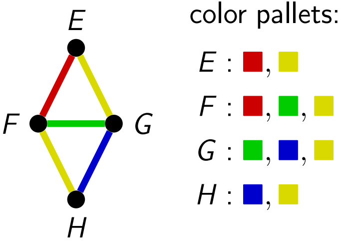


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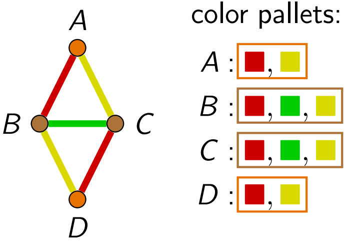


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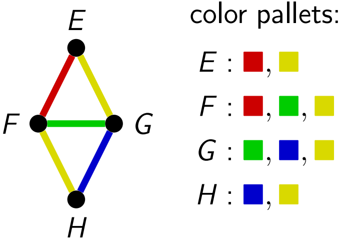


**GOOD**

It makes sense only if a graph contains no more than one isolated vertex and no isolated edges – 'to be nice property'.

# Vertex-distinguishing Edge Coloring

**Conjecture (Burris & Schelp)**

Every nice graph with $n$ vertices has vertex-distinguishing edge coloring by $n+1$ colors.

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**Theorem (Bazgan, Harkat-Benhamdine, Li, Woźniak, 99')**

Every nice graph with $n$ vertices has vertex-distinguishing edge coloring by $n+1$ colors.

# Vertex-distinguishing Edge Coloring

**Theorem**

*Every $d$-regular graph on $n$ vertices, which has a perfect matching, has a vertex-distinguishing edge coloring using at most*

$$K \sqrt[d]{n}$$

*colors, where* $K = 4\frac{d}{d-1} \sqrt[d]{\frac{1}{2}(d-1)!\,(d-1)}$.

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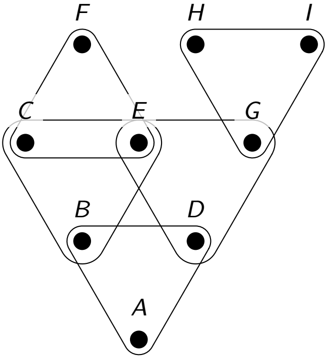
## Corollary

Each cubic graph with $n$ vertices, which has a perfect matching, is vertex-distinguishing edge coloring by $6\sqrt[3]{2n}$ colors.

# Harmonious Coloring of Hypergraph



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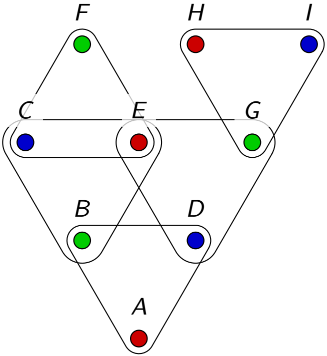
strong coloring of hypergraph – vertex coloring such that no two vertices of the same edge have the same color



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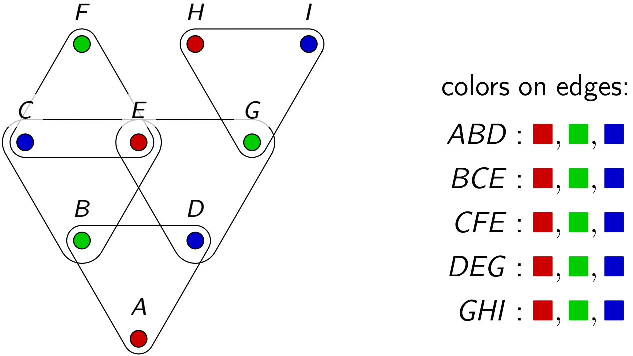

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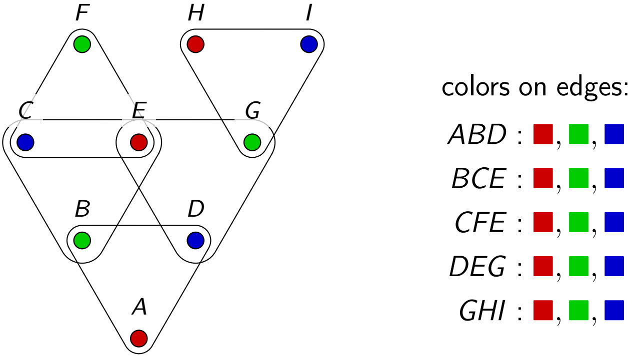
strong coloring of hypergraph – vertex coloring such that no two vertices of the same edge have the same color

harmonious coloring of hypergraph – strong coloring such that different edges have different palettes of colors



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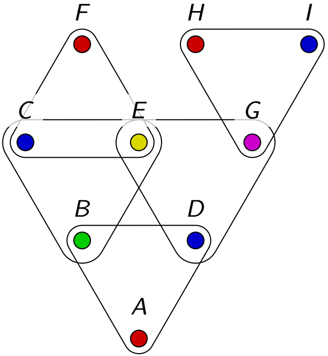



colors on edges:

*ABD* : ■, ■, ■

*BCE* : ■, ■, ■

*CFE* : ■, ■, ■

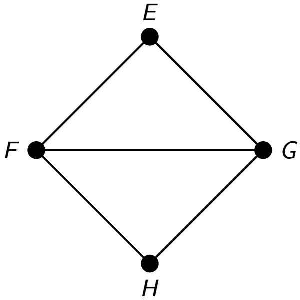
*DEG* : ■, ■, ■

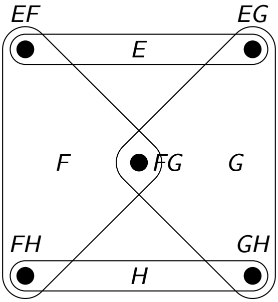
*GHI* : ■, ■, ■

# Vertex-distinguishing Edge Coloring as Harmonious Coloring

For a nice graph $G$ we can define hypergraph $H$, where:

- each edge in $G$ is a vertex in $H$,
- for each vertex $v$ in $G$ the set of edges incident to $v$ in $G$ form edge in $H$.



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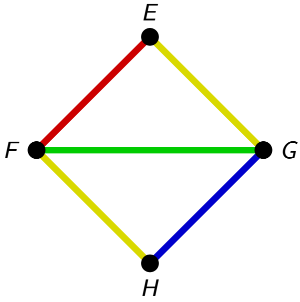


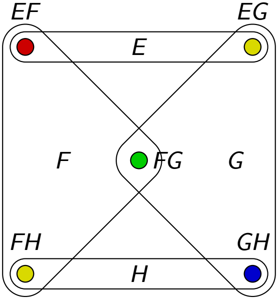

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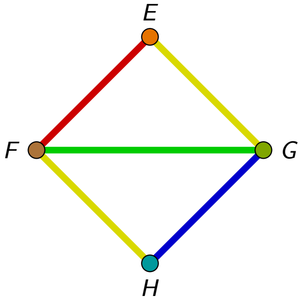


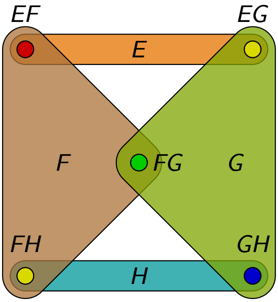

# Harmonious Coloring of Hypergraphs

**Theorem**

*Every $k$-uniform hypergraph $H$ with $m$ edges and maximum degree $\Delta$ satisfies*

$$h(H) \leqslant \frac{k}{k-1} \sqrt[k]{\Delta(k-1)k!m} + f(k, \Delta),$$

*where* $f(k, \Delta) = 1 + \Delta^2 + (k-1)\Delta + \sum_{i=2}^{k-1} \frac{i}{i-1} \sqrt[i]{(i-1)i! \frac{(k-1)\Delta^2}{k-i}}$.

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- Harmonious chromatic number of 2-uniform hypergraphs (simple graphs) is bounded by:

  $2\sqrt{2\Delta m} + \Delta^2 + \Delta$ (our), $2\sqrt{2\delta m} + 2\delta\Delta - 2\Delta$ (E&M 94').

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- Harmonious chromatic number of 3-uniform hypergraphs is bounded by:

$$3\sqrt[3]{3\Delta m} + \Delta^2 + 6\Delta.$$

# Entropy Compression

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*A constructive proof of the general Lovász Local Lemma.*

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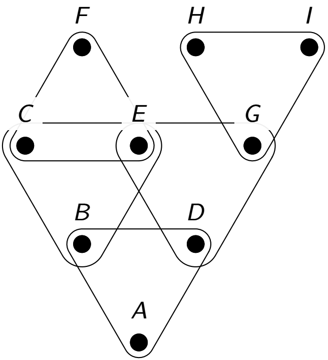
2013: **Esperet, Parreau**

*Acyclic edge-coloring using entropy compression.*

# Idea of the proof




colors on edges:

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sequence of colors : 

compressed sequence :

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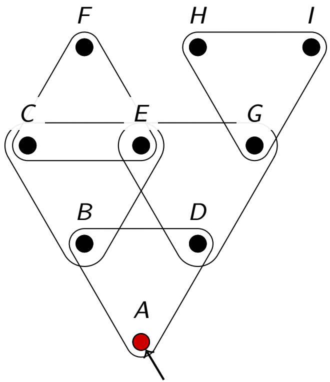


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compressed sequence : #

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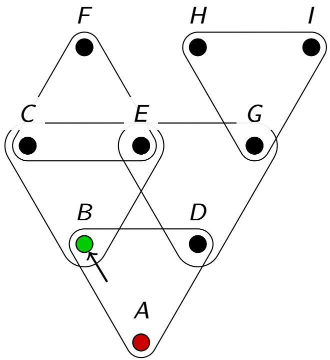


colors on edges:

*ABD* : ■, ■

*BCE* : ■

*CEF* :

*DEG* :

*GHI* :

sequence of colors : ■, ■, ■, ■, ■, ■, ■, ■, ■, ■, ■, ■, . . .

compressed sequence : #, #

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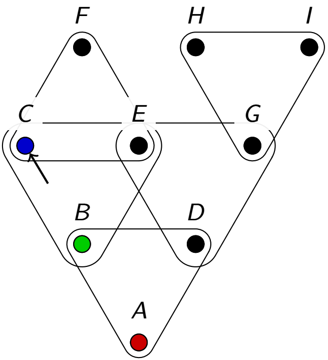


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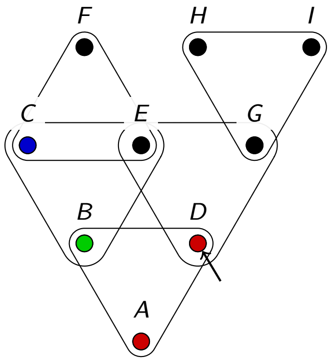


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compressed sequence : #, #, #, #

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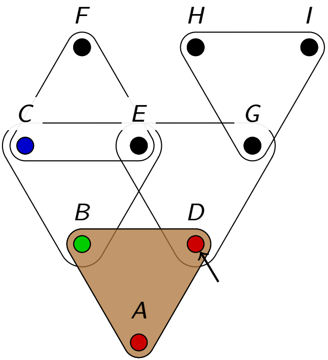


colors on edges:

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sequence of colors : ■, ■, ■, ■, ■, ■, ■, ■, ■, ■, ■, ■, . . .

compressed sequence : $\#, \#, \#, \#, [D \to ABD \to A]$

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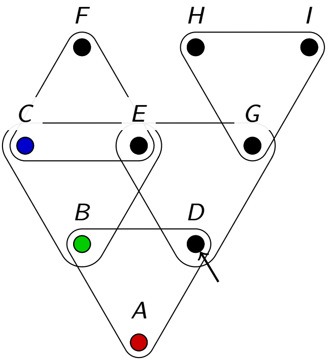


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sequence of colors : ■, ■, ■, ■, ■, ■, ■, ■, ■, ■, ■, ■, ...

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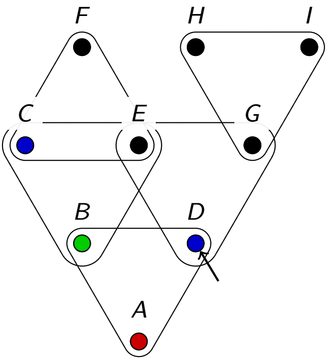


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sequence of colors : ■, ■, ■, ■, [■], ■, ■, ■, ■, ■, ■, ■, . . .

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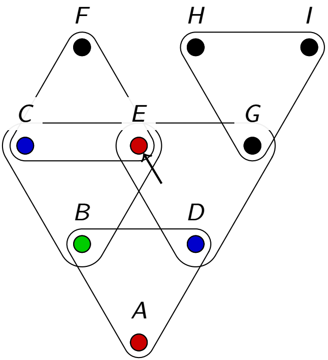


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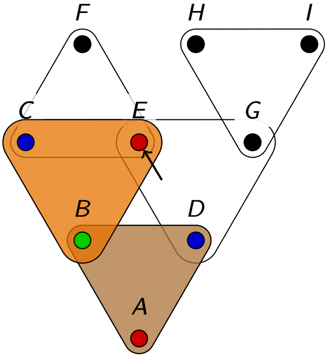


colors on edges:

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sequence of colors : , , , , , , , , , , , , …

compressed sequence : $\#, \#, \#, \#, [D \to ABD \to A], \#, \#,$
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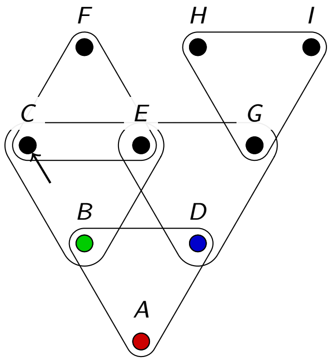


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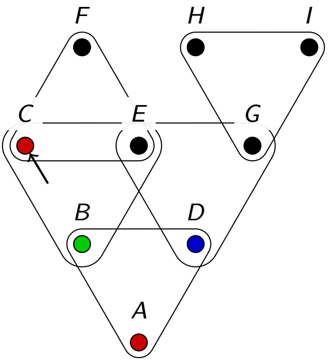


colors on edges:

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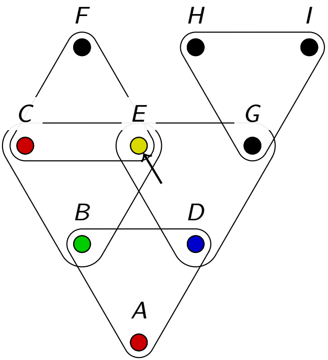
$DEG$ : 🟦

$GHI$ :

sequence of colors : 🟥, 🟩, 🟦, 🟥, 🟦, 🟥, [🟥], 🟨, 🟦, 🟩, 🟨, 🟥, . . .

compressed sequence : $\#, \#, \#, \#, [D \to ABD \to A], \#, \#,$
$[E \to CBE \to B \to ABD], \#$

# Idea of the proof




colors on edges:

$ABD$ : ■, ■, ■

$BCE$ : ■, ■, ■

$CEF$ : ■, ■

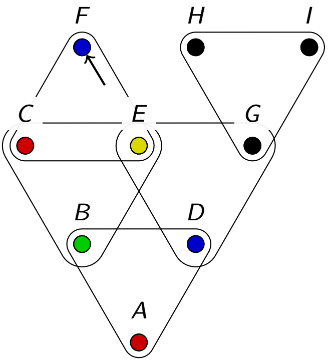
$DEG$ : ■, ■

$GHI$ :

sequence of colors : ■, ■, ■, ■, ■, ■, ■, ■, ■, ■, ■, ■, ...

compressed sequence : $\#, \#, \#, \#, [D \to ABD \to A], \#, \#,$
$[E \to CBE \to B \to ABD], \#, \#$

# Idea of the proof




colors on edges:

$ABD$ : ■, ■, ■

$BCE$ : ■, ■, ■

$CEF$ : ■, ■, ■

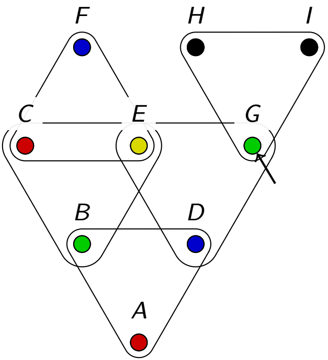
$DEG$ : ■, ■

$GHI$ :

sequence of colors : ■, ■, ■, ■, ■, ■, ■, ■, ■, ■, ■, ■, . . .

compressed sequence : $\#, \#, \#, \#, [D \to ABD \to A], \#, \#,$
$[E \to CBE \to B \to ABD], \#, \#, \#$

# Idea of the proof




colors on edges:

$ABD$ : ■, ■, ■

$BCE$ : ■, ■, ■

$CEF$ : ■, ■, ■

$DEG$ : ■, ■, ■

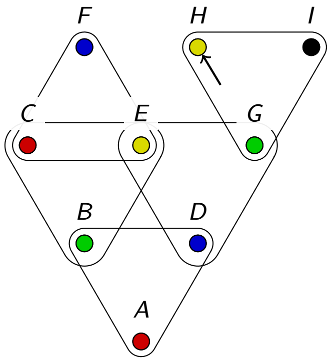
$GHI$ : ■

sequence of colors : ■, ■, ■, ■, ■, ■, ■, ■, ■, ■, ■, ■, . . .

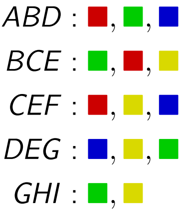
compressed sequence : $\#, \#, \#, \#, [D \to ABD \to A], \#, \#,$

$[E \to CBE \to B \to ABD], \#, \#, \#, \#$

# Idea of the proof



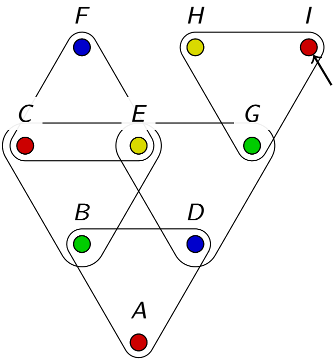

colors on edges:



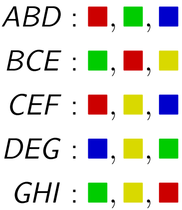

sequence of colors : 

compressed sequence : $\#, \#, \#, \#, [D \to ABD \to A], \#, \#,$
$[E \to CBE \to B \to ABD], \#, \#, \#, \#,$
$\#$

# Idea of the proof



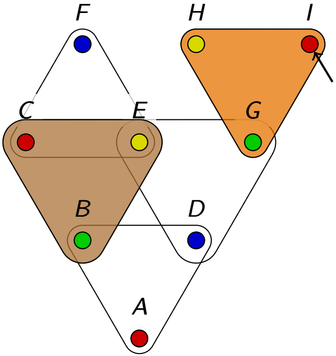

colors on edges:



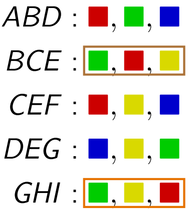

sequence of colors :  , . . .

compressed sequence : $\#, \#, \#, \#, [D \to ABD \to A], \#, \#,$
$[E \to CBE \to B \to ABD], \#, \#, \#, \#,$
$\#, \#$

# Idea of the proof



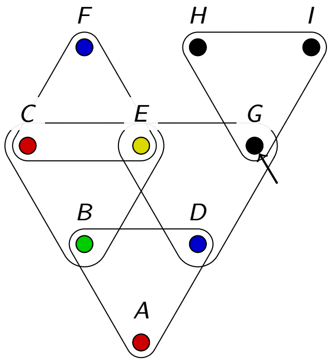

colors on edges:



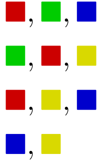

sequence of colors :

compressed sequence : #, #, #, #, $[D \to ABD \to A]$, #, #,
$[E \to CBE \to B \to ABD]$, #, #, #, #,
#, #, $[BCE]$

# Idea of the proof




colors on edges:



$ABD$ :

$BCE$ :

$CEF$ :

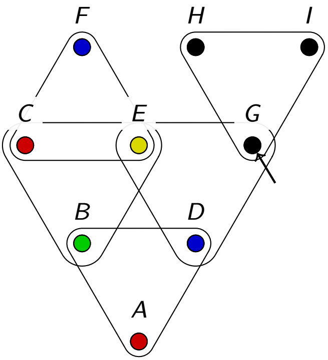
$DEG$ :

$GHI$ :

sequence of colors :  , . . .

compressed sequence : $\#, \#, \#, \#, [D \to ABD \to A], \#, \#,$
$[E \to CBE \to B \to ABD], \#, \#, \#, \#,$
$\#, \#, [BCE]$

# Idea of the proof




colors on edges:

$ABD$ : ■, ■, ■

$BCE$ : ■, ■, ■

$CEF$ : ■, ■, ■

$DEG$ : ■, ■

$GHI$ :

sequence of colors : ■, ■, ■, ■, ■, ■, ■, ■, ■, ■, ■, ■, . . .

compressed sequence : $\#, \#, \#, \#, [D \to ABD \to A], \#, \#,$
$[E \to CBE \to B \to ABD], \#, \#, \#, \#,$
$\#, \#, [BCE], \ldots$