Individual security and network design

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Motivation

- Networks often face external threats in form of strategic or random attacks
- The attacks can be prevented by protecting selected nodes of the network
- Decentralizing protection decisions leads to inefficiencies
- Can these inefficiencies be minimized by properly choosing the network topology?
Networks

- $G = (V, E)$: undirected graph over $V$, $V$ – set of nodes, $|V| = n$, $E$ – set of edges over $V$, $E \subseteq \{ij : i, j \in V\}$
- Given $U \subseteq V$, $E[U] = \{ij \in E : i, j \in U\}$, $G[U] = (U, E[U])$: subgraph induced by $U$
- $\mathcal{G}(U)$: set of all graphs over $U$
- Given $X \subseteq N$, $G - X = G[V \setminus X]$
- Component in $G$ is a maximal set of nodes $C \subseteq V$ s.t. for all $i, j \in C$, $i$ and $j$ are connected
- $\mathcal{C}(G)$: the set of all components in $G$
Network value function

- (Network) value function

\[ \Phi : \bigcup_{U \subseteq V} G(U) \rightarrow \mathbb{R} \]

assigns numeric value \( \Phi(G) \) to network \( G \in \bigcup_{U \subseteq V} G(U) \)

- We consider value functions of the form

\[ \Phi(G) = \sum_{C \in \mathcal{C}(G)} f(|C|), \]

where \( f \) is strictly increasing, strictly convex and \( f(0) = 0 \)

- Example 1: \( f(x) = x^2 \) (Metcalfe’s law)
- Example 2: \( f(x) = 2^x - 1 \) (≈ Reed’s law)
Defence and attack

- Defence is perfect: defended nodes cannot be removed
- Using defence has cost $c > 0$
- Given network $G$ and set of defended nodes $\Delta \subseteq V$, $G - \Delta$ is called **attack network**
- Attack targets a single node $x \in V$
- This removes set of nodes

$$E_x(G|\Delta) = \begin{cases} \emptyset & \text{if } x \in \Delta \\ C_x(G - \Delta) & \text{otherwise} \end{cases}$$
Defence and attack
Defence and attack

\[ g \]
Defence and attack

\(g, E_x(g|\Delta)\)
Defence and attack

$g - \Delta$
Defence and attack

\[ g - E_x(g|\Delta) \]
Exogenous network

- It is known that when the network is fixed, decentralization of defence decisions leads to inefficiencies
- Both underprotection and overprotection are possible in equilibrium of network defence game
- What happens when the designer can choose the network prior to network defence game?
The game

- There are \( n + 2 \) players: the designer (\( D \)), the nodes (\( V \)) and the adversary (\( A \))

- The game has three stages:
  1. \( D \) chooses network \( G \in \mathcal{G}(V) \)
  2. Nodes choose simultaneously whether to protect or not, this determines the set of defended nodes \( \Delta \)
  3. \( A \) chooses a node \( x \in V \) to attack

- The attack spreads which leads to residual network \( G - E_x(G|\Delta) \)
Preferences and payoffs: Designer

Given network $G \in \mathcal{G}(V)$, the set of defended nodes $\Delta \subseteq V$ and attack $x \in V$, payoff to $D$ is

$$\Pi^D(G, \Delta, x) = \Phi(G - E_x(G|\Delta)) - c|\Delta|$$
Given network $G \in \mathcal{G}(V)$, the set of defended nodes $\Delta \subseteq V$ and attack $x \in V$, payoff to node $i \in V$ is

$$\Pi^i(G, \Delta, x) = \begin{cases} 
\frac{f(|C_i(G - E_x(G|\Delta))|)}{|C_i(G - E_x(G|\Delta))|} - [x \in \Delta] \cdot c, & \text{if } i \neq x \\
0, & \text{otherwise}
\end{cases}$$
Preferences and payoffs: Adversary

- Given network $G \in \mathcal{G}(V)$, the set of defended nodes $\Delta \subseteq V$ and attack $x \in V$, payoff to $A$ is

$$\Pi^A(G, \Delta, x) = -\phi(G - E_x(G|\Delta))$$
Proposition (1)

Let \((G, \Delta)\) be optimal defended network chosen by the designer and let

- If \(0 < c < c_1(n)\), then \(g\) is connected and \(\Delta = V\)
- If \(c_1(n) < c < c_2(n)\), then \(g\) is a star and \(\Delta = \{i\}\), where \(i\) is the centre of \(g\)
- If \(c_2(n) < c\), then \(g\) has \(q^* - 1\) components of size \(\lfloor \frac{n}{q^* - 1} \rfloor\) and one of size \(n \mod (q^* - 1)\), \(\Delta = \emptyset\)
Optimal network with decentralized defence

Let $\Gamma(G)$ denote the nodes-adversary subgame ensuing after network $G$ is chosen.

Lemma (1)

For any $G$ and $c$, $\Gamma(G)$ has an equilibrium in pure strategies.

Let $\mathcal{E}(G, c)$ denote the set of all equilibria of $\Gamma(G)$. 
Optimal network with decentralized defence

Definition
Given $G \in \mathcal{G}(n)$, equilibrium $(\Delta, x) \in \mathcal{E}(G, c)$ is welfare maximising iff

$$\Pi^D(G, \Delta, x) = \max_{(\Delta', x') \in \mathcal{E}(G, c)} \Pi^D(G, \Delta', x')$$

and $(\Delta, x) \in \mathcal{E}(G, c)$ is welfare minimising iff

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Optimal network with decentralized defence

- There are three main reasons to inefficiencies due to decentralization:
  1. Underprotection, due to positive externalities (e.g. $f(x) = x^2$, $n < c < (n-1)^2$ and centre of a star network)
  2. Overprotection, due to negative externalities (e.g. $f(x) = x^2$, $2n-1 < n-1 < n$ and all nodes of star network)
  3. Coordination on wrong equilibria (e.g. connected network and all- or no-nodes protect)
Optimal network with decentralized defence

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3. **Coordination on wrong equilibria** (e.g. connected network and all- or no-nodes protect)

Can the designer mitigate these problems by carefully choosing the network topology?

How well can s/he do?
Optimal networks under welfare maximising equilibria

- What networks should be chosen assuming that the nodes and A choose a welfare maximising equilibrium in their subgame?
- The only problem is the range of costs \((c_1(n), c_2(n))\) with centrally protected star being first best
- We need to avoid overprotection \(\rightarrow\) reduce incentives to protect
Optimal networks under welfare maximising equilibria

Example: \( f(x) = x^2, \ n = 96, \ c = 30 \)
Optimal networks under welfare minimising equilibria

- What networks should be chosen assuming that the nodes and \textbf{A} choose a welfare minimising equilibrium in their subgame?
- There is no problem for high costs
- Splitting the network still works for intermediate costs (the equilibrium outcome is unique on the split network)
- When costs of protection are low we need to avoid the coordination problem
- Equilibrium uniqueness can be obtained by properly choosing network topology
Optimal networks under welfare minimising equilibria

Definition \((k\text{-critical node})\)

Given a connected network \(G \in \mathcal{G}(V)\) node \(i \in V\) is \(k\)-critical iff

\[
\max_{C \in \mathcal{C}(G \setminus \{i\})} |C| = k
\]

- \(k\)-critical node, if it protects, can secure at least \(n - k\) nodes from being removed
Optimal networks under welfare minimizing equilibria

\( f(x) = x^2, \ n = 19, \ 9 < c < 10 \)
Lemma (7)

Let $G \in \mathcal{G}(V)$ be a connected network. If all nodes protect in every equilibrium of $\Gamma(G)$, then $G$ has a $k$-critical node where $k$ satisfies

$$\frac{f(n-k)}{n-k} > c$$

- Existence of $k$-critical node with $\frac{f(n-k)}{n-k} > c$ is a necessary but not sufficient condition for full protection in every equilibrium of $\Gamma(G)$
Optimal networks under welfare minimising equilibria

\[ f(x) = x^2, \quad n = 19, \quad 9 < c < 10 \]
Lemma (8)

Let $G \in \mathcal{G}(V)$ be a connected network and $k$ satisfy $\frac{f(n-k)}{n-k} > c$. If for all $i \in V$, either $i$ is $k$-critical or $i$ has link to a $k$-critical node, then all nodes protect in every equilibrium of $\Gamma(g)$. 

Optimal networks under welfare minimising equilibria

\[ f(x) = x^2, \quad n = 19, \quad 9 < c < 10 \]
Optimal networks under welfare minimising equilibria

\[ f(x) = x^2, \quad n = 19, \quad 9 < c < 10 \]
Interaction design

- The presented problem is an example of a more general idea.
- Given a type of local interactions (game) can we enforce desired outcomes by properly choosing connections between nodes?