Individual security and network design

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Motivation

- Networks often face external threats in form of strategic or random attacks
- The attacks can be prevented by protecting selected nodes of he network
- Decentralizing protection decisions leads to inefficiencies
- Can these inefficiencies be minimized by properly choosing the network topology?

Networks

- $G = (V, E)$: undirected graph over V, V – set of nodes, $|V| = n$, E – set of edges over V, $E \subseteq \{ij : i, j \in V\}$
- Given $U \subseteq V$, $E[U] = \{ij \in E : i, j \in U\}$, $G[U] = (U, E[U])$: subgraph induced by U
- $\mathcal{G}(U)$: set of all graphs over U
- Given $X \subseteq N$, $G - X = G[V \setminus X]$
- Component in G is a maximal set of nodes $C \subseteq V$ s.t. for all $i, j \in C$, i and j are connected
- $\mathcal{C}(G)$: the set of all components in G

Network value function

- **(Network) value function**

$$\Phi : \bigcup_{U \subseteq V} \mathcal{G}(U) \to \mathbb{R}$$

  assigns numeric value $\Phi(G)$ to network $G \in \bigcup_{U \subseteq V} \mathcal{G}(U)$
- We consider value functions of the form

$$\Phi(G) = \sum_{C \in \mathcal{C}(G)} f(|C|),$$

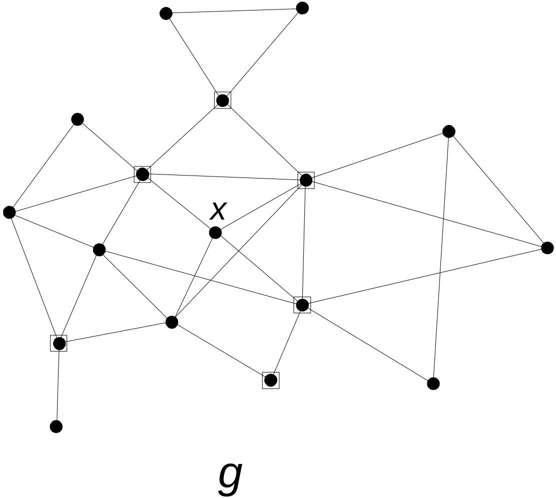
  where f is strictly increasing, strictly convex and $f(0) = 0$
  - Example 1: $f(x) = x^2$ (Metcalfe's law)
  - Example 2: $f(x) = 2^x - 1$ ($\approx$ Reed's law)

Defence and attack

- Defence is perfect: defended nodes cannot be removed
- Using defence has cost $c > 0$
- Given network G and set of defended nodes $\Delta \subseteq V$, $G - \Delta$ is called **attack network**
- Attack targets a single node $x \in V$
- This removes set of nodes

$$E_x(G|\Delta) = \begin{cases} \varnothing & \text{if } x \in \Delta \\ C_x(G - \Delta) & \text{otherwise} \end{cases}$$

Defence and attack



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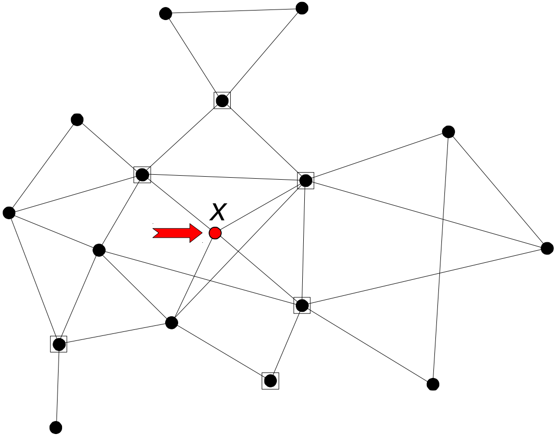

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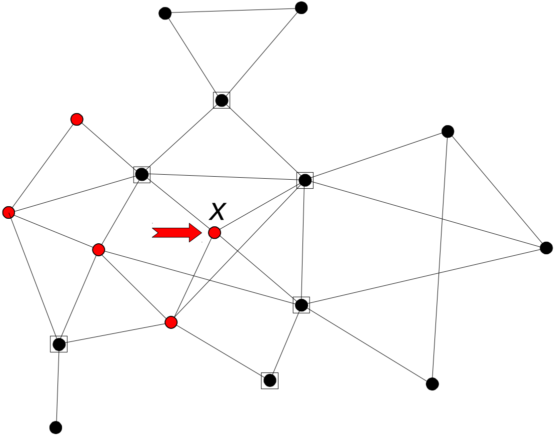
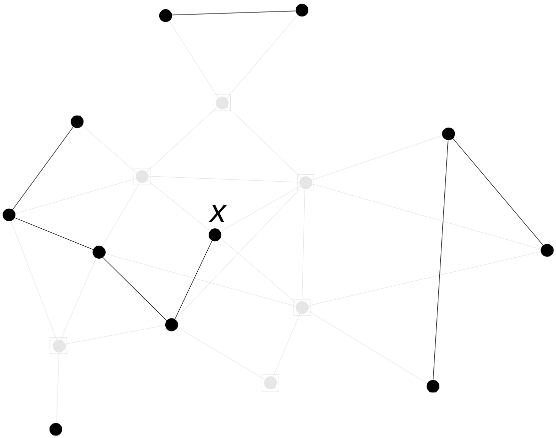
Defence and attack



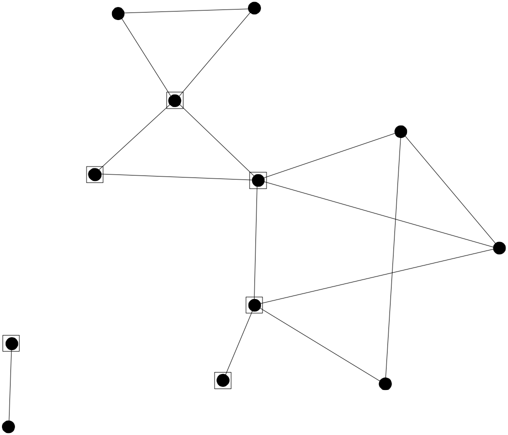
$$g, E_x(g|\Delta)$$

Defence and attack



$$g - \Delta$$

Defence and attack



$$g - E_X(g|\Delta)$$

Exogenous network

- It is known that when the network is fixed, decentralization of defence decisions leads to inefficiencies
- Both underprotection and overprotection are possible in equilibrium of network defence game
- What happens when the designer can choose the network prior to network defence game?

The game

- There are $n+2$ players: the designer (**D**), the nodes (V) and the adversary (**A**)
- The game has three stages:
  1. **D** chooses network $G \in \mathcal{G}(V)$
  2. Nodes choose simultaneously whether to protect or not, this determines the set of defended nodes Δ
  3. **A** chooses a node $x \in V$ to attack
- The attack spreads which leads to **residual network** $G - E_x(G|\Delta)$

Preferences and payoffs: Designer

- Given network $G \in \mathcal{G}(V)$, the set of defended nodes $\Delta \subseteq V$ and attack $x \in V$, payoff to **D** is

$$\Pi^{\mathbf{D}}(G, \Delta, x) = \Phi(G - E_x(G|\Delta)) - c|\Delta|$$

Preferences and payoffs: Nodes

- Given network $G \in \mathcal{G}(V)$, the set of defended nodes $\Delta \subseteq V$ and attack $x \in V$, payoff to node $i \in V$ is

$$\Pi^i(G, \Delta, x) = \begin{cases} \frac{f(|C_i(G - E_x(G|\Delta))|)}{|C_i(G - E_x(G|\Delta))|} - [x \in \Delta] \cdot c, & \text{if } i \neq x \\ 0, & \text{otherwise} \end{cases}$$

Preferences and payoffs: Adversary

- Given network $G \in \mathcal{G}(V)$, the set of defended nodes $\Delta \subseteq V$ and attack $x \in V$, payoff to **A** is

$$\Pi^{\mathbf{A}}(G, \Delta, x) = -\Phi(G - E_x(G|\Delta))$$

First best

Proposition (1)

Let (G, Δ) *be optimal defended network chosen by the designer and let*

- *If* $0 < c < c_1(n)$, *then* g *is connected and* $\Delta = V$
- *If* $c_1(n) < c < c_2(n)$, *then* g *is a star and* $\Delta = \{i\}$, *where* i *is the centre of* g
- *If* $c_2(n) < c$, *then* g *has* $q^* - 1$ *components of size* $\lfloor \frac{n}{q^*-1} \rfloor$ *and one of size* $n \bmod (q^* - 1)$, $\Delta = \varnothing$

Optimal network with decentralized defence

- Let $\Gamma(G)$ denote the nodes-adversary subgame ensuing after network G is chosen

Lemma (1)

For any G and c, $\Gamma(G)$ has an equilibrium in pure strategies

- Let $\mathcal{E}(G, c)$ denote the set of all equilibria of $\Gamma(G)$

Optimal network with decentralized defence

Definition

Given $G \in \mathcal{G}(n)$, equilibrium $(\Delta, x) \in \mathcal{E}(G, c)$ is **welfare maximising** iff

$$\Pi^{\mathbf{D}}(G, \Delta, x) = \max_{(\Delta', x') \in \mathcal{E}(G, c)} \Pi^{\mathbf{D}}(G, \Delta', x')$$

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  2. **Overprotection**, due to negative externalities (e.g. $f(x) = x^2$, $\frac{2n-1}{n-1} < c < n-1$ and all nodes of star network)

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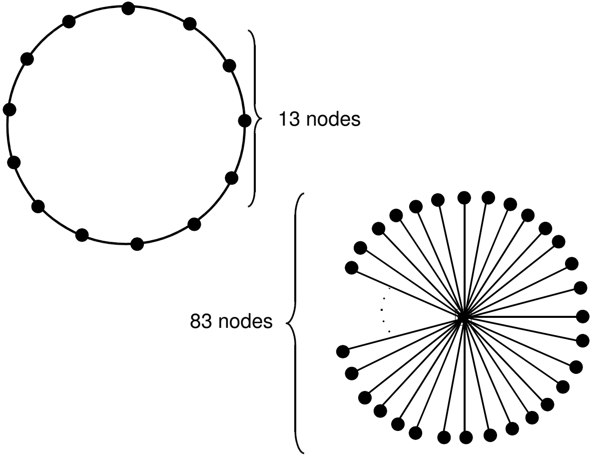
  3. **Coordination on wrong equilibria** (e.g. connected network and all- or no-nodes protect)
- Can the designer mitigate these problems by carefully choosing the network topology?
- How well can s/he do?

Optimal networks under welfare maximising equilibria

- What networks should be chosen assuming that the nodes and **A** choose a welfare maximising equilibrium in their subgame?
- The only problem is the range of costs $(c_1(n), c_2(n))$ with centrally protected star being first best
- We need to avoid overprotection $\longrightarrow$ reduce incentives to protect

Optimal networks under welfare maximising equilibria

- Example: $f(x) = x^2$, $n = 96$, $c = 30$



Optimal networks under welfare minimising equilibria

- What networks should be chosen assuming that the nodes and **A** choose a welfare minimising equilibrium in their subgame?
- There is no problem for high costs
- Splitting the network still works for intermediate costs (the equilibrium outcome is unique on the split network)
- When costs of protection are low we need to avoid the coordination problem
- Equilibrium uniqueness can be obtained by properly choosing network topology

Optimal networks under welfare minimising equilibria

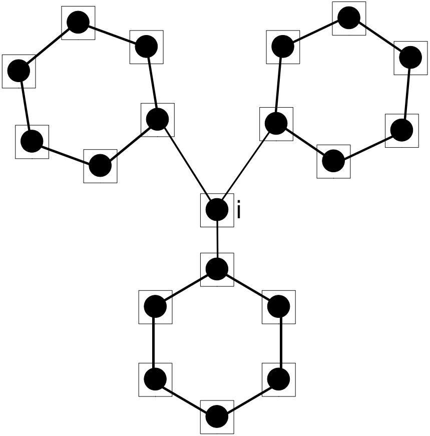
Definition (k-critical node)

Given a connected network $G \in \mathcal{G}(V)$ node $i \in V$ is k-critical iff $\max_{C \in \mathcal{C}(G-\{i\})} |C| = k$

- k-critical node, if it protects, can secure at least $n - k$ nodes from being removed

Optimal networks under welfare minimising equilibria

$f(x) = x^2$, $n = 19$, $9 < c < 10$



Optimal networks under welfare minimising equilibria

Lemma (7)

Let $G \in \mathcal{G}(V)$ be a connected network. If all nodes protect in every equilibrium of $\Gamma(G)$, then G has a k-critical node where k satisfies $\frac{f(n-k)}{n-k} > c$

- Existence of k-critical node with $\frac{f(n-k)}{n-k} > c$ is a necessary but not sufficient condition for full protection in every equilibrium of $\Gamma(G)$

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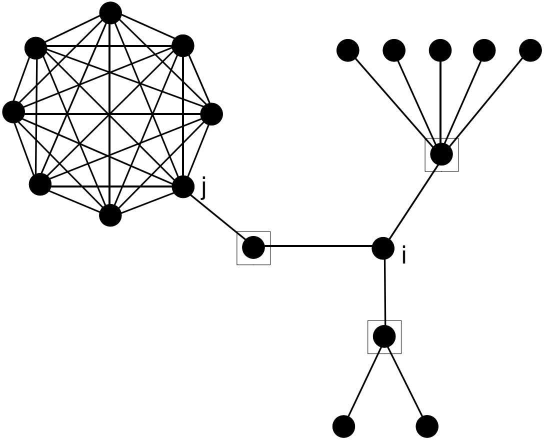

Optimal networks under welfare minimising equilibria

Lemma (8)

Let $G \in \mathcal{G}(V)$ be a connected network and k satisfy $\frac{f(n-k)}{n-k} > c$. If for all $i \in V$, either i is k-critical or i has link to a k-critical node, then all nodes protect in every equilibrium of $\Gamma(g)$

Optimal networks under welfare minimising equilibria

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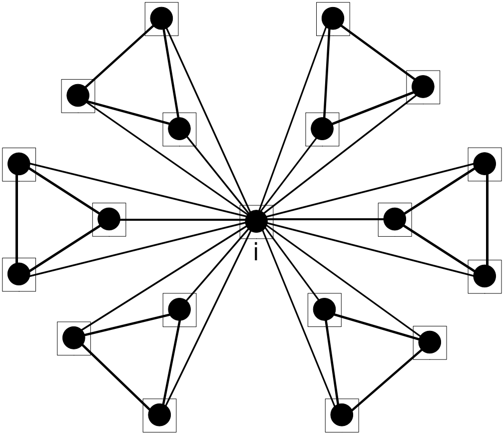

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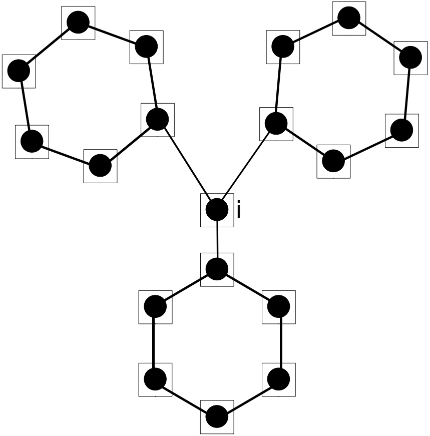

Interaction design

- The presented problem is an example of a more general idea
- Given a type of local interactions (game) can we enforce desired outcomes by properly choosing connections between nodes?