Undecidability and complexity in fragments of first-order intuitionistic logic

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Prenex normal form

In classical logic:

Every formula is classically equivalent to one of the form:

$$Q_1 x_1 Q_2 x_2 \ldots Q_k x_k.\, \mathit{Body}(x_{,} x_2, \ldots, x_k),$$

where *Body* has no quantifiers.

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In intuitionistic logic:

The prenex fragment is decidable in Pspace.

The language we study

To make things simpler, we consider first-order formulas

- with universal quantifiers and implications;
- without function symbols.

This fragment is known to be undecidable.

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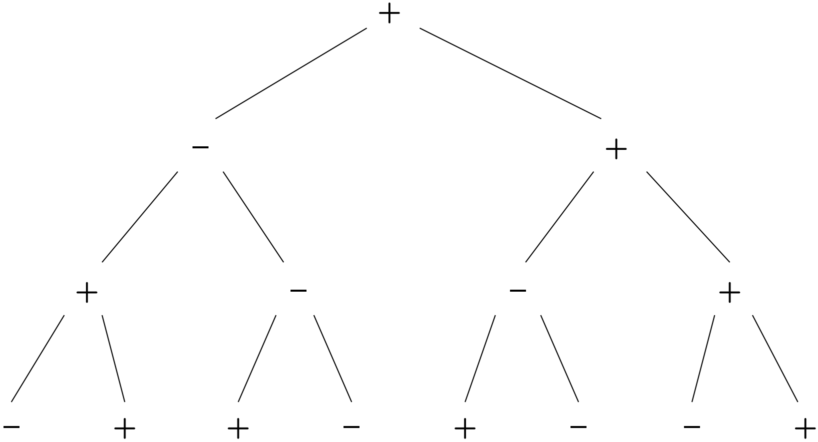
Positive and Negative



∘

Mints Hierarchy

Π_1 – All quantifiers at positive positions.

Σ_1 – All quantifiers at negative positions.

Π_2 – One alternation: some negative quantifiers in scope of some positive ones.

Σ_2 – One alternation: some positive quantifiers in scope of some negative ones.

And so on.

Examples

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$((\forall x\, P(x)) \to Q) \to R$ is a Π_1 (positive) formula;

$(\forall x\, (\forall y\, P(x) \to \forall z\, S(y, z))) \to R$ is a Σ_2 formula.

∘

Σ_1 decision problem

Negative formulas are of shape

$$\varphi_1 \to \cdots \to \varphi_n \to \sigma,$$

where σ is an atomic formula and $\varphi_1, \ldots, \varphi_n$ are positive.

A Σ_1 decision problem can be seen as

$$\varphi_1, \ldots, \varphi_n \vdash \sigma$$

(Derive an atom from positive assumptions.)

Similarly for Π_1, Σ_2, Π_2

Complexity of Mints Hierarchy

- Π_2 and Σ_2 are undecidable (even Δ_1).
- Π_1 is **co-2-NExptime**-hard
  (conjecture: super-elementary)
- Σ_1:
  - **Expspace**-complete in general;
  - **co-Nexptime**-complete with monadic predicates.

Upper bound for Σ_1

Proofs in Σ_1 may use universal assumptions, which can be instantiated. But introducing new variables is of no use.

If there is a proof then there is one where all eigenvariables are free variables of the original formula φ.

Such proof can be constructed in exponential space.
(Every subformula of φ of size n has at most n^n instances.)

Mechanics of lower bound for Σ_1

Assumptions:

$\forall xyz\,(P(x,y,z,1)\to P(x,y,z,0))$,

$\forall xy\,(P(x,y,1,0)\to P(x,y,0,1))$,

$\forall x\,(P(x,1,0,0)\to P(x,0,1,1))$,

$P(1,0,0,0)\to P(0,1,1,1)$.

Goal: $P(0,0,0,0)$

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$\forall x\,(P(x, 1, 0, 0) \to P(x, 0, 1, 1))$,

$P(1, 0, 0, 0) \to P(0, 1, 1, 1)$.

Goal: a few more steps. . .

Mechanics of lower bound for Σ_1

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Goal:

The proof search is as difficult as rewriting **0000** into **1111**.

Automata?

$$P(1,0,0,0) \to P(0,1,1,1).$$

is like the transition from $P(0,1,1,1)$ to $P(1,0,0,0)$.

and

$$\forall xyz\,(P(x,y,z,1) \to P(x,y,z,0)),$$

is like a template rule that makes it possible to represent exponentially many states.

Exponential control + tree structure = Expspace

Approaching Π_1

Observe that to prove

$$\Gamma, (\forall x. P(x) \to R) \to Q \vdash Q$$

we can use $(\forall x. P(x) \to R) \to Q$ and reduce the original problem to

$$\Gamma, (\forall x. P(x) \to R) \to Q \vdash \forall x. P(x) \to R$$

and further to

$$\Gamma, (\forall x. P(x) \to R) \to Q, P(x') \vdash R$$

where x' is a fresh variable.

Σ_2 and Π_2

- Actually $\mathrm{Cl}(\Pi_1 \cup \Sigma_1)$ is enough.
- Put $\forall x.(P(x) \to \text{loop}) \to \text{loop}$ at positive position.
- This makes it possible to generate arbitrary many eigenvariables.
- Use otherwise the construction for monadic version of Σ_1.

Conclusions

- The decision problem for Σ_1 is Exspspace-complete.
  Important: only free variables in the original formula are used.
  Important: the number of targets is exponential.
- The decision problem for Σ_1 with at most unary predicates is co-Nexptime-complete.
  Important: small number of targets.
- The decision problem for Π_1 is 2-coNExptime-hard, (maybe superelementary).
  Important: hereditarily finite sets can be represented.
- Σ_2 and Π_2 are undecidable.
  Important: actually $\Delta_1 = \mathsf{Cl}(\Sigma_1 \cup \Pi_1)$ is.