Asymptotic properties of combinatory logic

Maciej Bendkowski

joined work with Katarzyna Grygiel and Marek Zaionc

Theoretical Computer Science
Jagiellonian University

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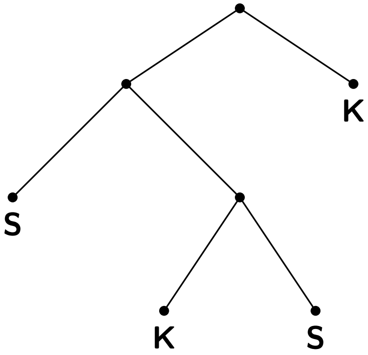


Figure: $\mathbf{S}(\mathbf{KS})\mathbf{K} \in \mathtt{CL}$

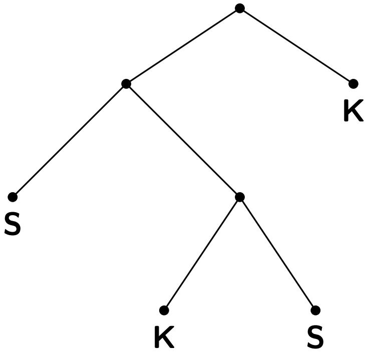
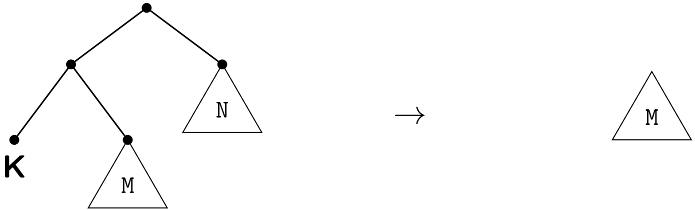


Figure: $\mathbf{S}(\mathbf{KS})\mathbf{K} \in \mathtt{CL}$

$$\mathbf{S}, \mathbf{K} \in \mathtt{CL} \qquad M, N \in \mathtt{CL} \implies (MN) \in \mathtt{CL}$$

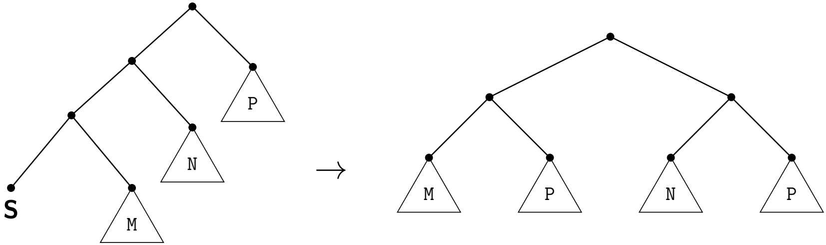
K reduction rule

$\mathbf{K}MN \to M$ for arbitrary $M, N \in \mathtt{CL}$



S reduction rule

$\mathbf{S}MNP \to MP(NP)$ for arbitrary $M, N, P \in \mathtt{CL}$



- If there exists no combinator Q such that $P \to Q$, then P is in **normal form**.

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- If there exists a finite sequence $P_1, \ldots, P_k$ such that $P_1 \to P_2 \to \ldots \to P_k$ and P_k is in normal form, then P_1 is **weakly normalizing (normalizable)**.

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- If there exists a finite sequence $P_1, \ldots, P_k$ such that $P_1 \to P_2 \to \ldots \to P_k$ and P_k is in normal form, then P_1 is **weakly normalizing (normalizable)**.
- If there does not exist an infinite sequence $P_1, P_2 \ldots$ such that $P_1 \to P_2 \to \ldots$ then P_1 is **strongly normalizing**.

Research problems:

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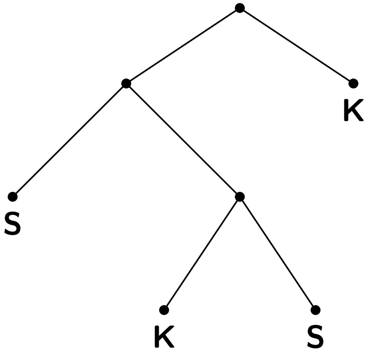

Figure: $|\mathbf{S}(\mathbf{KS})\mathbf{K}| = 3$

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$$\left|\mathtt{CL}_n\right| = 2^{n+1} C_n = \frac{2^{n+1}}{n+1}\binom{2n}{n} < \infty$$

where C_n is the n-th Catalan number.

Let $A \subseteq B \subseteq \texttt{CL}$. The *asymptotic density* of class A in B, is defined as

$$\mu(\frac{A}{B}) := \lim_{n\to\infty} \frac{|A_n|}{|B_n|} \quad \left(\text{assuming} \quad \frac{0}{0} = 1\right)$$

provided that the limit exists.

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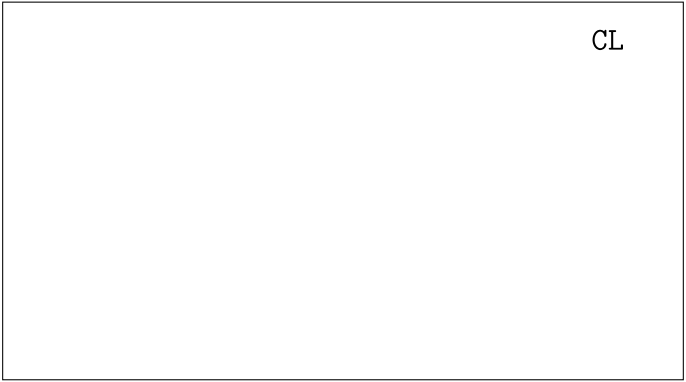
provided that the limit exists.
Moreover, we define

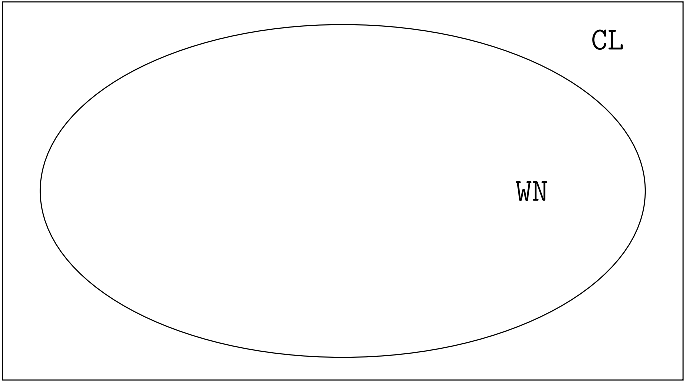
$$\mu^+(\frac{A}{B}) := \limsup_{n\to\infty} \frac{|A_n|}{|B_n|} \qquad \mu^-(\frac{A}{B}) := \liminf_{n\to\infty} \frac{|A_n|}{|B_n|}$$

well-defined for arbitrary $A \subseteq B \subseteq \mathtt{CL}$.


CL

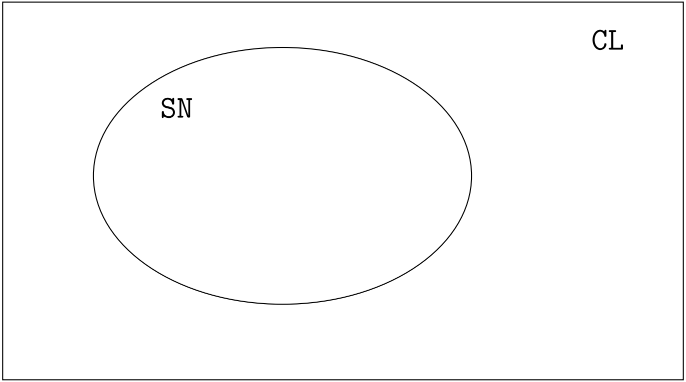
Theorem (BGZ, 2015)

$0 < \mu(\frac{\mathtt{WN}}{\mathtt{CL}}) < 1$ provided that the limit exists.



Theorem (David, G, Kozik, Raffalli, Theyssier, Z, 2013)

$\mu(\frac{\mathtt{SN}}{\mathtt{CL}}) = 0.$



Theorem (BGZ, 2015)

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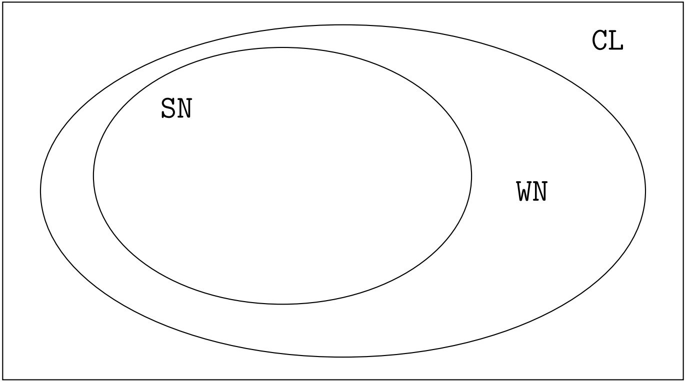
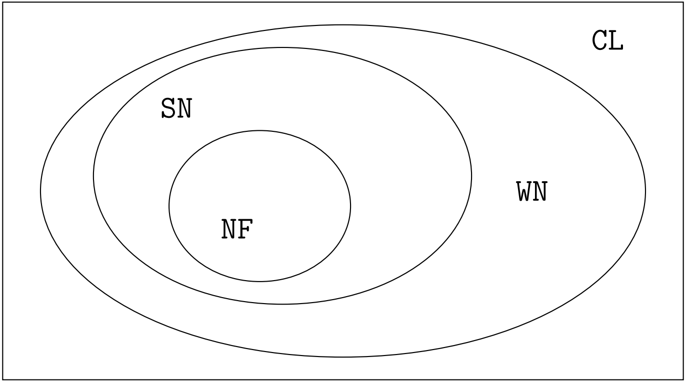


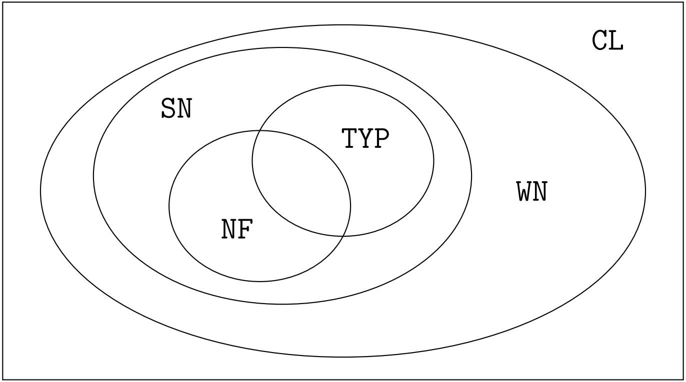
Theorem (BGZ, 2015)

$\mu\left(\frac{\mathtt{NF}}{\mathtt{SN}}\right) = 0.$



Theorem (BGZ, 2015)

$\mu(\frac{\mathtt{NF} \cap \mathtt{TYP}}{\mathtt{NF}}) = 0.$



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- If so, what is the value of $\mu(\frac{\texttt{WN}}{\texttt{CL}})$?
- What is the value of $\mu(\frac{\texttt{TYP}}{\texttt{SN}})$?

References:

- David R., Grygiel K., Kozik J., Raffalli C., Theyssier G., Zaionc M., Asymptotically almost all λ-terms are strongly normalizing, Logical Methods in Computer Science Vol. 9, 2013.
- Bendkowski M., Grygiel K., Zaionc M., Asymptotic properties of combinatory logic [accepted to TAMC 2015].

Questions and answers