Optimal decremental connectivity in planar graphs

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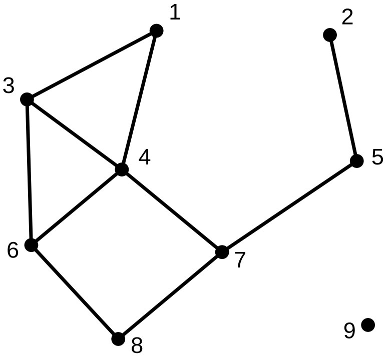
Dynamic connectivity

Input: an undirected graph G

Update: Add/remove an edge.

Query: Are vertices u and v connected with a path?

Example



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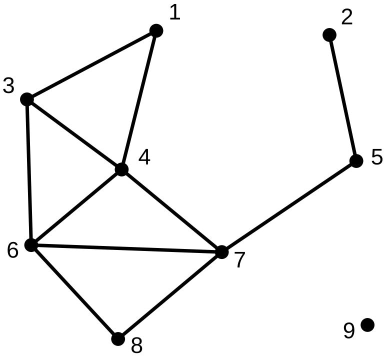
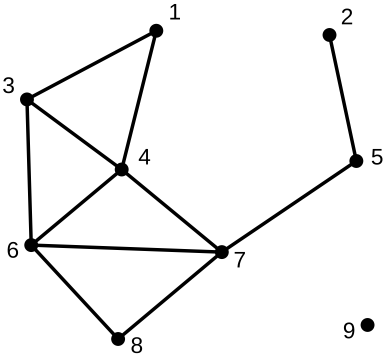


add(6,7)

Example



`query(1,6)?`

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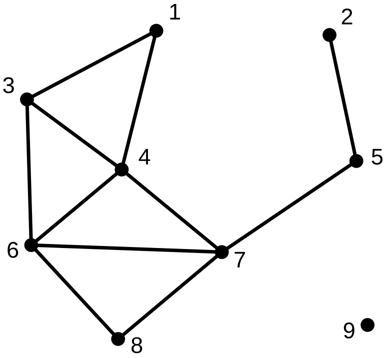
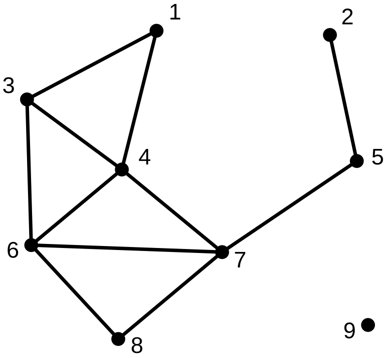


query(1,6)? YES

Example



`remove(5,7)`

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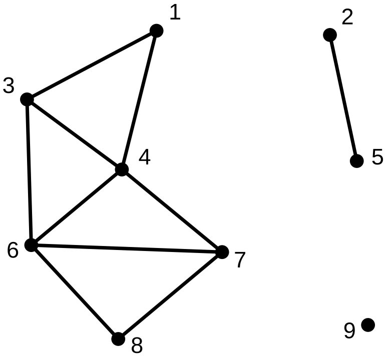

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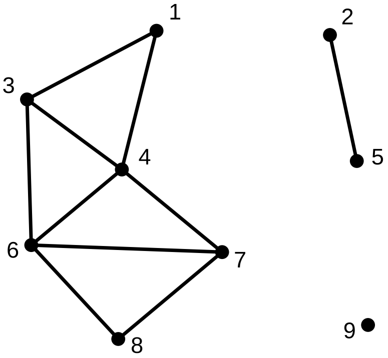
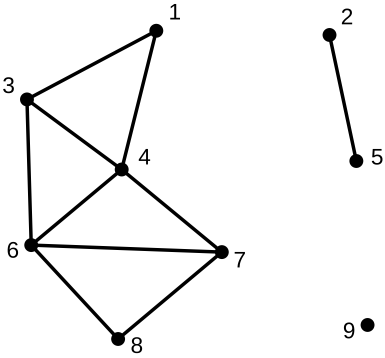


`query(2,8)?`

Example



query(2,8)? NO

Three variants

Dynamic graph problems (in particular connectivity) come in three variants:

- *incremental* – edges can only be added
- *decremental* – edges can only be deleted
- *fully dynamic* – both edge insertions and deletions are allowed

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Updates and queries in $O(\alpha(n))$ amortized time.

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Decremental connectivity (Thorup 1999)

For $m = \Omega(n(\log n \log\log n)^2)$ any sequence of deletions can be handled in $O(m \log n)$ time. Queries answered in $O(1)$ time.

Fully dynamic connectivity in planar graphs

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Updates and queries in $O(\log n)$ amortized time, provided that the embedding does not change over time.

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Corollary

There exists a decremental connectivity algorithm for planar graphs that handles all updates in $O(n \log n)$ time.

Dynamic connectivity in trees

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Both in general graphs and trees there exists a decremental connectivity algorithm, which is faster then the fully dynamic one.

Our result

Theorem

There exists a decremental connectivity algorithm for planar graphs that supports updates in $O(n)$ total time and answers queries in constant time.

Overview

1. Introduction
   - Dynamic connectivity
   - Related work
   - Our result
2. Our algorithms
   - Critical deletions
   - O(n log n) time algorithm
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   - O(n) time algorithm

Preliminaries

- We work with a planar graph $G = (V, E)$, subject to edge deletions
- G has n vertices and $O(n)$ edges
- We may assume that G is initially connected, and the degree of every vertex is at most 3

Critical deletion

Definition

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We first show how to detect critical deletions.

Theorem (Euler's formula)

Let $G = (V, E)$ be a planar graph, $v = |V|, e = |E|$, f be the number of faces, and c be the number of connected components of G. Then

$$v - e + f = c + 1$$

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- Thus, we need to maintain the number of faces of G.

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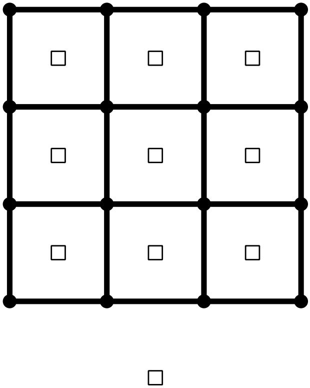
- Thus, we need to maintain the number of faces of G.
- When an edge is deleted, we merge faces on both sides.

Example



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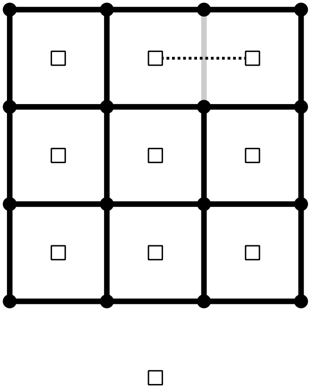

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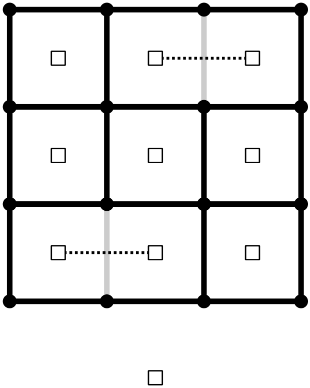

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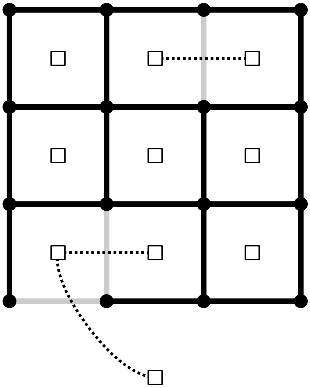

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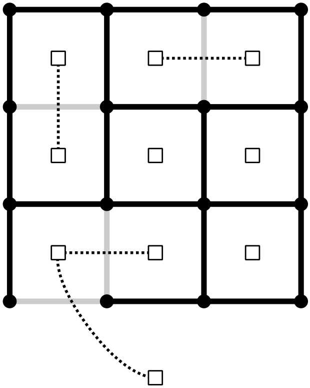

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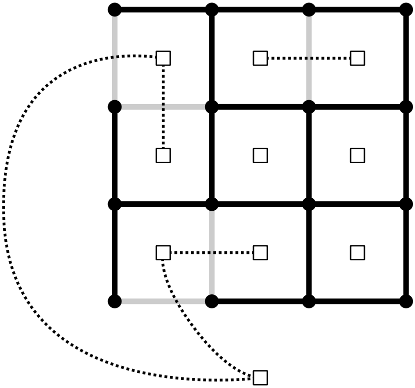

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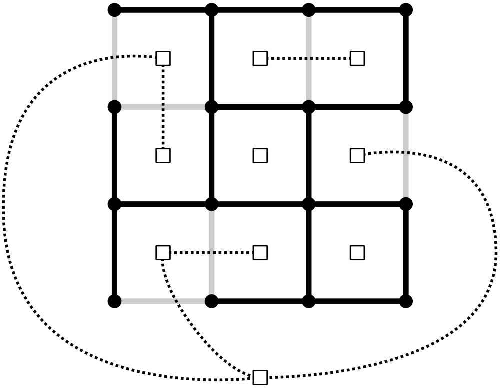

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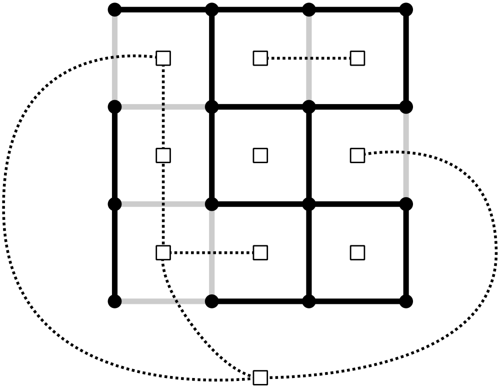

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We can detect critical deletions with $O(n)$ overhead.

O(n log n) time algorithm

The algorithm, for every vertex v maintains a *cc-identifier*, that is a unique identifier of its connected component. Thus

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query(u, w):
return true iff cc-identifier[u] = cc-identifier[w]
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How to update the cc-identifiers after an edge deletion?

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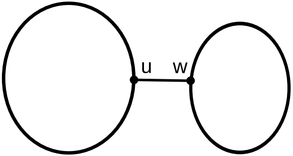
After an edge *uw* is deleted:

- If the deletion is not critical, do nothing.
- Otherwise, start two parallel DFS searches from *u* and *w*



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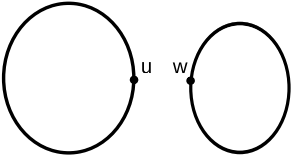

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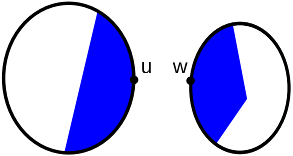

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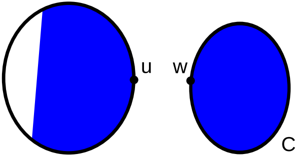

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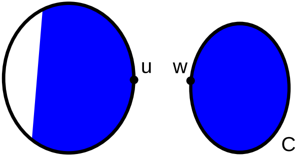

C is the *smaller* among the two new connected components.

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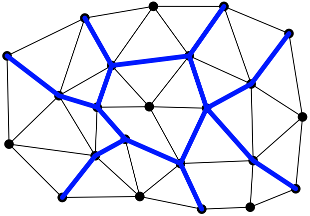
Analysis

- The running time is proportional to the number of changes of cc-identifiers.
- Every time a vertex changes its cc-identifier, the size of its connected component halves.
- All updates are processed in $O(n \log n)$ time.

O(n log log n) time algorithm

To obtain a faster algorithm we use an *r-division*.



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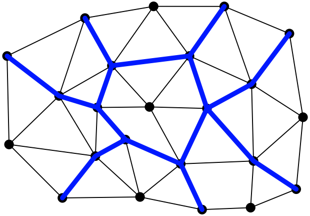

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We use $r = \log^2 n$.

O(n log log n) time algorithm

Overall idea

- Build an r-division for $r = \log^2 n$.
- Run the $O(n \log n)$ algorithm inside every region.
- Use a similar idea to maintain cc-identifiers of boundary vertices (there are $O(n/\log n)$ of them).

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- Each such graph can be encoded as a binary string of length $s^2 = o(\log n)$, which fits in a machine word.
- We can precompute connected components of all small graphs in $o(n)$ time!

Main result

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Thank you!

Questions?