The game of overlaps

Jarosław Grytczuk **Karol Kosiński** Michał Zmarz

Jagiellonian University

31 January 2015

Definitions

A - alphabet

Repetition: xx , where $x \in A^+$

Overlap: $axaxa$, where $a \in A, x \in A^*$

Examples:

$A = \{0, 1, 2\}$

01210121	is a repetition
11	is a repetition
201120112	is an overlap
000	is an overlap

THEORETICAL COMPUTER SCIENCE Jagiellonian University

Definitions

A - alphabet

Repetition:	xx	, where $x \in A^+$
Overlap:	$axaxa$	, where $a \in A, x \in A^*$

Examples:

$A = \{0, 1, 2\}$

01210121	is a repetition
11	is a repetition
201120112	is an overlap
000	is an overlap

Sparse overlap-free game

Ben: I want an overlap!

Ann: No overlaps!

Sparse overlap-free game

Ben: I want an overlap!

Ann: No overlaps!

					0									...	

Sparse overlap-free game

Ben: I want an overlap!

Ann: No overlaps!

					0	1								...	

Sparse overlap-free game

Ben: I want an overlap!
Ann: No overlaps!

					0	1		1						...	

THEORETICAL COMPUTER SCIENCE Jagiellonian University

Sparse overlap-free game

Ben: I want an overlap!

Ann: No overlaps!

					0	1	2	1						···	

Sparse overlap-free game

Ben: I want an overlap!

Ann: No overlaps!

					0	1	2	1		1				$\cdots$	

Sparse overlap-free game

Ben: I want an overlap!
Ann: No overlaps!

					0	1	2	1	0	1				...	

Sparse overlap-free game

Ben: I want an overlap!
Ann: No overlaps!

					0	1	2	1	0	1		1		$\cdots$	

Sparse overlap-free game

Ben: I want an overlap!
Ann: No overlaps!

					0	1	2	1	0	1	2	1		$\cdots$	

Sparse overlap-free game

Ben: I want an overlap!
Ann: No overlaps!

				1	0	1	2	1	0	1	2	1		$\cdots$	

Sparse overlap-free game

Ben: I want an overlap!
Ann: No overlaps!

				1	0	1	2	1	0	1	2	1		$\cdots$	

Ben wins

THEORETICAL COMPUTER SCIENCE Jagiellonian University

Main result

Theorem (Grytczuk, Kosiński, Zmarz)

There exists a winning strategy with finite description for **Ann** *that allows her to win the sparse overlap-free game of any even length over a 4-letter alphabet (provided* **Ben** *starts the play).*

The result above is optimal – **Ben** can win the game that is played on a 3-letter alphabet in just 5 moves!

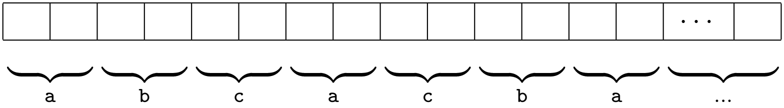
Main result

Theorem (Grytczuk, Kosiński, Zmarz)

There exists a winning strategy with finite description for **Ann** *that allows her to win the sparse overlap-free game of any even length over a 4-letter alphabet (provided* **Ben** *starts the play).*

The result above is optimal – **Ben** can win the game that is played on a 3-letter alphabet in just 5 moves!

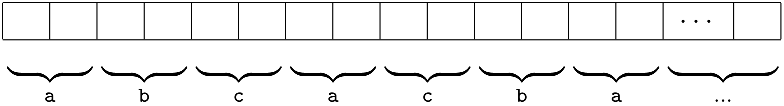
Ann's strategy



$$t = \underline{\mathtt{abcacba}}\mathtt{bcbacabcacbacabcb}\ldots = \lim_{i\to\infty}\phi^i(\mathtt{a})$$

$$\phi = \begin{cases} \mathtt{a} & \to \mathtt{abc} \\ \mathtt{b} & \to \mathtt{ac} \\ \mathtt{c} & \to \mathtt{b} \end{cases}$$

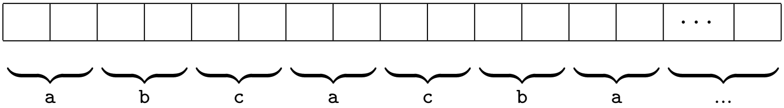
Ann's strategy



$$t = \underline{\mathtt{abcacba}}\mathtt{bcbacabcacbacabcb}\ldots = \lim_{\mathtt{i}\to\infty} \phi^{\mathtt{i}}(\mathtt{a})$$

$$\phi = \begin{cases} \mathtt{a} & \to \mathtt{abc} \\ \mathtt{b} & \to \mathtt{ac} \\ \mathtt{c} & \to \mathtt{b} \end{cases}$$

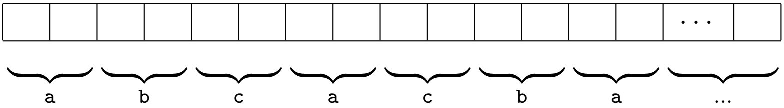
Ann's strategy



$$t = \underline{\textcolor{red}{\mathtt{a}}\mathtt{bcacba}}\mathtt{bcbacabcacbacabcb}\ldots = \lim_{\mathtt{i}\to\infty} \phi^{\mathtt{i}}(\textcolor{red}{\mathtt{a}})$$

$$\phi = \begin{cases} \mathtt{a} & \to \mathtt{abc} \\ \mathtt{b} & \to \mathtt{ac} \\ \mathtt{c} & \to \mathtt{b} \end{cases}$$

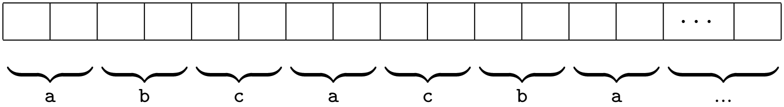
Ann's strategy



$$t = \underline{\mathtt{abcacba}}\mathtt{bcbacabcacbacabcb}\ldots = \lim_{\mathtt{i}\to\infty} \phi^{\mathtt{i}}(\mathtt{a})$$

$$\phi = \begin{cases} \mathtt{a} & \to \mathtt{abc} \\ \mathtt{b} & \to \mathtt{ac} \\ \mathtt{c} & \to \mathtt{b} \end{cases}$$

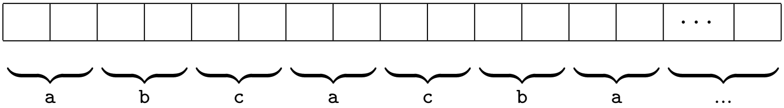
Ann's strategy



$$t = \underline{\color{red}{\mathtt{abc}}\color{green}{\mathtt{acb}}\mathtt{a}}\mathtt{bcbacabcacbacabcb}\ldots = \lim_{\mathtt{i}\to\infty} \phi^{\mathtt{i}}(\mathtt{a})$$

$$\phi = \begin{cases} \mathtt{a} & \to \mathtt{abc} \\ \mathtt{b} & \to \mathtt{ac} \\ \mathtt{c} & \to \mathtt{b} \end{cases}$$

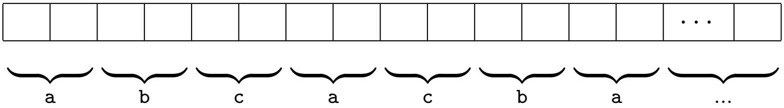
Ann's strategy



$$t = \underline{\color{red}{\texttt{abcacb}}\color{green}{\texttt{a}}}\color{green}{\texttt{bcbac}}\texttt{abcacbacabcb}\ldots = \lim_{\texttt{i}\to\infty} \phi^{\texttt{i}}(\texttt{a})$$

$$\phi = \begin{cases} \texttt{a} & \to \texttt{abc} \\ \texttt{b} & \to \texttt{ac} \\ \texttt{c} & \to \texttt{b} \end{cases}$$

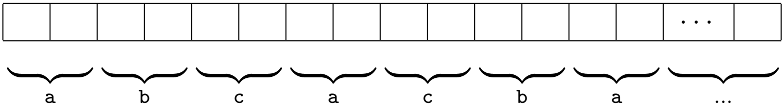
Ann's strategy



$$t = \underline{\mathtt{abcacba}}\mathtt{bcbacabcacbacabcb}\ldots = \lim_{i \to \infty} \phi^{i}(\mathtt{a})$$

$$\phi = \begin{cases} \mathtt{a} & \to \mathtt{abc} \\ \mathtt{b} & \to \mathtt{ac} \\ \mathtt{c} & \to \mathtt{b} \end{cases}$$

Ann's strategy



$$t = \underline{\texttt{abcacba}}\texttt{bcbacabcacbacabcb}\ldots = \lim_{\texttt{i}\to\infty} \phi^{\texttt{i}}(\texttt{a})$$

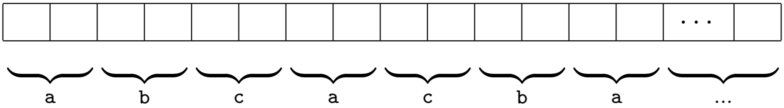
$$\phi = \begin{cases} \texttt{a} & \to \texttt{abc} \\ \texttt{b} & \to \texttt{ac} \\ \texttt{c} & \to \texttt{b} \end{cases}$$

t – an infinite Thue word without repetitions over a 3-letter alphabet

Ann's strategy

Ben: I want an overlap!

Ann: No overlaps!



$$\mathtt{a} \rightarrow \{01, 12, 23, 30\}$$

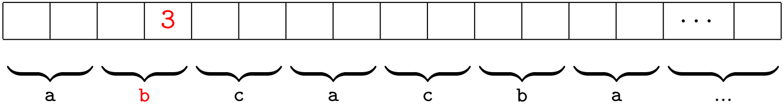
$$\mathtt{b} \rightarrow \{02, 13, 20, 31\}$$

$$\mathtt{c} \rightarrow \{03, 10, 21, 32\}$$

Ann's strategy

Ben: I want an overlap!

Ann: No overlaps!

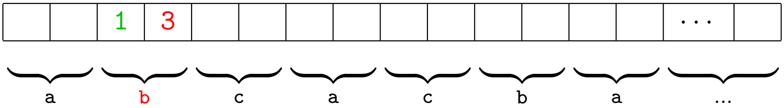


$$\begin{aligned} a &\rightarrow \{01, 12, 23, 30\} \\ b &\rightarrow \{02, 13, 20, 31\} \\ c &\rightarrow \{03, 10, 21, 32\} \end{aligned}$$

Ann's strategy

Ben: I want an overlap!

Ann: No overlaps!



$$\mathtt{a} \rightarrow \{01, 12, 23, 30\}$$

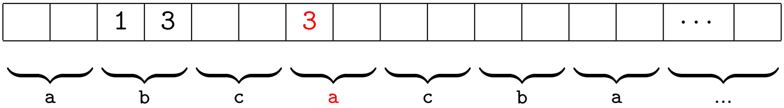
$$\mathtt{b} \rightarrow \{02, 13, 20, 31\}$$

$$\mathtt{c} \rightarrow \{03, 10, 21, 32\}$$

Ann's strategy

Ben: I want an overlap!

Ann: No overlaps!



$$\begin{aligned} a &\rightarrow \{01, 12, 23, 30\} \\ b &\rightarrow \{02, 13, 20, 31\} \\ c &\rightarrow \{03, 10, 21, 32\} \end{aligned}$$

Ann's strategy

Ben: I want an overlap!

Ann: No overlaps!

		1	3			3	0							...	
a		b		c		a		c		b		a		...	

$$\mathtt{a} \rightarrow \{01, 12, 23, 30\}$$

$$\mathtt{b} \rightarrow \{02, 13, 20, 31\}$$

$$\mathtt{c} \rightarrow \{03, 10, 21, 32\}$$

Ann's strategy

Ben: I want an overlap!
Ann: No overlaps!

		1	3		1	3	0							$\cdots$

a b c a c b a …

$$\mathtt{a} \to \{01, 12, 23, 30\}$$
$$\mathtt{b} \to \{02, 13, 20, 31\}$$
$$\mathtt{c} \to \{03, 10, 21, 32\}$$

THEORETICAL COMPUTER SCIENCE Jagiellonian University

Ann's strategy

Ben: I want an overlap!
Ann: No overlaps!

		1	3	2	1	3	0							⋯	
a		b		c		a		c		b		a		...	

$$\mathtt{a} \rightarrow \{01, 12, 23, 30\}$$
$$\mathtt{b} \rightarrow \{02, 13, 20, 31\}$$
$$\mathtt{c} \rightarrow \{03, 10, 21, 32\}$$

Why does it work?

Suppose that an overlap $axaxa$ exists as a subword and $m = |x|$.

	$\cdots$	a	x_1	x_2	$\cdots$	x_m	a	x_1	x_2	$\cdots$	x_m	a	$\cdots$	

- m is **odd**

$$\underbrace{a x_1}_{r_1} \underbrace{x_2 x_3}_{r_2} \underbrace{x_4 x_5}_{r_3} \underbrace{x_6 x_7}_{r_4} \underbrace{a x_1}_{r_1} \underbrace{x_2 x_3}_{r_2} \underbrace{x_4 x_5}_{r_3} \underbrace{x_6 x_7}_{r_4} a$$

or

$$a \underbrace{x_1 x_2}_{s_1} \underbrace{x_3 x_4}_{s_2} \underbrace{x_5 x_6}_{s_3} \underbrace{x_7 a}_{s_4} \underbrace{x_1 x_2}_{s_1} \underbrace{x_3 x_4}_{s_2} \underbrace{x_5 x_6}_{s_3} \underbrace{x_7 a}_{s_4}$$

rr or ss – a subword of t

t – an infinite Thue word without repetitions

$\Rightarrow$ contradiction

THEORETICAL COMPUTER SCIENCE Jagiellonian University

Why does it work?

Suppose that an overlap $axaxa$ exists as a subword and $m = |x|$.

	$\cdots$	a	x_1	x_2	$\cdots$	x_m	a	x_1	x_2	$\cdots$	x_m	a	$\cdots$	

- m is **odd**

$$\underbrace{a x_1}_{r_1} \underbrace{x_2 x_3}_{r_2} \underbrace{x_4 x_5}_{r_3} \underbrace{x_6 x_7}_{r_4} \underbrace{a x_1}_{r_1} \underbrace{x_2 x_3}_{r_2} \underbrace{x_4 x_5}_{r_3} \underbrace{x_6 x_7}_{r_4} a \qquad \text{or}$$

$$a \underbrace{x_1 x_2}_{s_1} \underbrace{x_3 x_4}_{s_2} \underbrace{x_5 x_6}_{s_3} \underbrace{x_7 a}_{s_4} \underbrace{x_1 x_2}_{s_1} \underbrace{x_3 x_4}_{s_2} \underbrace{x_5 x_6}_{s_3} \underbrace{x_7 a}_{s_4}$$

rr or ss – a subword of t

t – an infinite Thue word without repetitions

$\Rightarrow$ contradiction

Why does it work?

Suppose that an overlap $axaxa$ exists as a subword and $m = |x|$.

	$\cdots$	a	x_1	x_2	$\cdots$	x_m	a	x_1	x_2	$\cdots$	x_m	a	$\cdots$	

- m is **odd**

$$\underbrace{a x_1}_{r_1}\underbrace{x_2 x_3}_{r_2}\underbrace{x_4 x_5}_{r_3}\underbrace{x_6 x_7}_{r_4}\underbrace{a x_1}_{r_1}\underbrace{x_2 x_3}_{r_2}\underbrace{x_4 x_5}_{r_3}\underbrace{x_6 x_7}_{r_4} a \qquad \text{or}$$

$$a \underbrace{x_1 x_2}_{s_1}\underbrace{x_3 x_4}_{s_2}\underbrace{x_5 x_6}_{s_3}\underbrace{x_7 a}_{s_4}\underbrace{x_1 x_2}_{s_1}\underbrace{x_3 x_4}_{s_2}\underbrace{x_5 x_6}_{s_3}\underbrace{x_7 a}_{s_4}$$

rr or ss – a subword of t

t – an infinite Thue word without repetitions

$\Rightarrow$ contradiction

Why does it work?

Suppose that an overlap $axaxa$ exists as a subword and $m = |x|$.

	...	a	x_1	x_2	...	x_m	a	x_1	x_2	...	x_m	a	...	

- m is **odd**

$$\underbrace{ax_1}_{r_1}\underbrace{x_2x_3}_{r_2}\underbrace{x_4x_5}_{r_3}\underbrace{x_6x_7}_{r_4}\underbrace{ax_1}_{r_1}\underbrace{x_2x_3}_{r_2}\underbrace{x_4x_5}_{r_3}\underbrace{x_6x_7}_{r_4}a \qquad \underline{\text{or}}$$

$$a\underbrace{x_1x_2}_{s_1}\underbrace{x_3x_4}_{s_2}\underbrace{x_5x_6}_{s_3}\underbrace{x_7a}_{s_4}\underbrace{x_1x_2}_{s_1}\underbrace{x_3x_4}_{s_2}\underbrace{x_5x_6}_{s_3}\underbrace{x_7a}_{s_4}$$

rr <u>or</u> ss – a subword of t

t – an infinite Thue word without repetitions

$\Rightarrow$ contradiction

Why does it work?

Suppose that an overlap $axaxa$ exists as a subword and $m = |x|$.

	...	a	x_1	x_2	...	x_m	a	x_1	x_2	...	x_m	a	...	

- m is **odd**

$$\underbrace{ax_1}_{r_1}\underbrace{x_2x_3}_{r_2}\underbrace{x_4x_5}_{r_3}\underbrace{x_6x_7}_{r_4}\underbrace{ax_1}_{r_1}\underbrace{x_2x_3}_{r_2}\underbrace{x_4x_5}_{r_3}\underbrace{x_6x_7}_{r_4}a \qquad \text{or}$$

$$a\underbrace{x_1x_2}_{s_1}\underbrace{x_3x_4}_{s_2}\underbrace{x_5x_6}_{s_3}\underbrace{x_7a}_{s_4}\underbrace{x_1x_2}_{s_1}\underbrace{x_3x_4}_{s_2}\underbrace{x_5x_6}_{s_3}\underbrace{x_7a}_{s_4}$$

rr or ss – a subword of t

t – an infinite Thue word without repetitions

$\Rightarrow$ contradiction

Why does it work?

Suppose that an overlap $axaxa$ exists as a subword and $m = |x|$.

	$\cdots$	a	x_1	x_2	$\cdots$	x_m	a	x_1	x_2	$\cdots$	x_m	a	$\cdots$	

- m is **odd**

$$\underbrace{ax_1}_{r_1}\underbrace{x_2x_3}_{r_2}\underbrace{x_4x_5}_{r_3}\underbrace{x_6x_7}_{r_4}\underbrace{ax_1}_{r_1}\underbrace{x_2x_3}_{r_2}\underbrace{x_4x_5}_{r_3}\underbrace{x_6x_7}_{r_4}a \qquad \text{or}$$

$$a\underbrace{x_1x_2}_{s_1}\underbrace{x_3x_4}_{s_2}\underbrace{x_5x_6}_{s_3}\underbrace{x_7a}_{s_4}\underbrace{x_1x_2}_{s_1}\underbrace{x_3x_4}_{s_2}\underbrace{x_5x_6}_{s_3}\underbrace{x_7a}_{s_4}$$

rr or ss – a subword of t

t – an infinite Thue word without repetitions

$\Rightarrow$ contradiction

THEORETICAL COMPUTER SCIENCE Jagiellonian University

Why does it work?

- m is **even**

$$\underbrace{a x_1}_{u_1}\underbrace{x_2 x_3}_{u_3}\underbrace{x_4 x_5}_{u_5}\underbrace{x_6 a}_{u_7}\underbrace{x_1 x_2}_{u_2}\underbrace{x_3 x_4}_{u_4}\underbrace{x_5 x_6}_{u_6} a$$

or

$$a \underbrace{x_1 x_2}_{u_2}\underbrace{x_3 x_4}_{u_4}\underbrace{x_5 x_6}_{u_6}\underbrace{a x_1}_{u_1}\underbrace{x_2 x_3}_{u_3}\underbrace{x_4 x_5}_{u_5}\underbrace{x_6 a}_{u_7}$$

$$a \xrightarrow{u_1} x_1 \xrightarrow{u_2} x_2 \xrightarrow{u_3} x_3 \xrightarrow{u_4} x_4 \xrightarrow{u_5} x_5 \xrightarrow{u_6} x_6 \xrightarrow{u_7} a$$

$$\mathrm{a} \longrightarrow \{0 \xrightarrow{\bar{\mathrm{a}}} 1, 1 \xrightarrow{\bar{\mathrm{a}}} 2, 2 \xrightarrow{\bar{\mathrm{a}}} 3, 3 \xrightarrow{\bar{\mathrm{a}}} 0\} \quad \Rightarrow \quad \bar{\mathrm{a}} \equiv +1 \pmod 4$$

$$\mathrm{b} \longrightarrow \{0 \xrightarrow{\bar{\mathrm{b}}} 2, 1 \xrightarrow{\bar{\mathrm{b}}} 3, 2 \xrightarrow{\bar{\mathrm{b}}} 0, 3 \xrightarrow{\bar{\mathrm{b}}} 1\} \quad \Rightarrow \quad \bar{\mathrm{b}} \equiv +2 \pmod 4$$

$$\mathrm{c} \longrightarrow \{0 \xrightarrow{\bar{\mathrm{c}}} 3, 1 \xrightarrow{\bar{\mathrm{c}}} 0, 2 \xrightarrow{\bar{\mathrm{c}}} 1, 3 \xrightarrow{\bar{\mathrm{c}}} 2\} \quad \Rightarrow \quad \bar{\mathrm{c}} \equiv +3 \pmod 4$$

$$value + \overline{u_1} + \overline{u_2} + \overline{u_3} + \overline{u_4} + \overline{u_5} + \overline{u_6} + \overline{u_7} \equiv value \pmod 4$$

$$\overline{u_1} + \overline{u_2} + \overline{u_3} + \overline{u_4} + \overline{u_5} + \overline{u_6} + \overline{u_7} \equiv 0 \pmod 4$$

Why does it work?

- m is **even**

$$\underbrace{a x_1}_{u_1}\underbrace{x_2 x_3}_{u_3}\underbrace{x_4 x_5}_{u_5}\underbrace{x_6 a}_{u_7}\underbrace{x_1 x_2}_{u_2}\underbrace{x_3 x_4}_{u_4}\underbrace{x_5 x_6}_{u_6} a \qquad \text{or}$$

$$a \underbrace{x_1 x_2}_{u_2}\underbrace{x_3 x_4}_{u_4}\underbrace{x_5 x_6}_{u_6}\underbrace{a x_1}_{u_1}\underbrace{x_2 x_3}_{u_3}\underbrace{x_4 x_5}_{u_5}\underbrace{x_6 a}_{u_7}$$

$$a \xrightarrow{u_1} x_1 \xrightarrow{u_2} x_2 \xrightarrow{u_3} x_3 \xrightarrow{u_4} x_4 \xrightarrow{u_5} x_5 \xrightarrow{u_6} x_6 \xrightarrow{u_7} a$$

$$\mathrm{a} \longrightarrow \{0 \xrightarrow{\bar{a}} 1, 1 \xrightarrow{\bar{a}} 2, 2 \xrightarrow{\bar{a}} 3, 3 \xrightarrow{\bar{a}} 0\} \quad \Rightarrow \quad \bar{\mathrm{a}} \equiv +1 \pmod 4$$

$$\mathrm{b} \longrightarrow \{0 \xrightarrow{\bar{b}} 2, 1 \xrightarrow{\bar{b}} 3, 2 \xrightarrow{\bar{b}} 0, 3 \xrightarrow{\bar{b}} 1\} \quad \Rightarrow \quad \bar{\mathrm{b}} \equiv +2 \pmod 4$$

$$\mathrm{c} \longrightarrow \{0 \xrightarrow{\bar{c}} 3, 1 \xrightarrow{\bar{c}} 0, 2 \xrightarrow{\bar{c}} 1, 3 \xrightarrow{\bar{c}} 2\} \quad \Rightarrow \quad \bar{\mathrm{c}} \equiv +3 \pmod 4$$

$$value + \overline{u_1} + \overline{u_2} + \overline{u_3} + \overline{u_4} + \overline{u_5} + \overline{u_6} + \overline{u_7} \equiv value \pmod 4$$

$$\overline{u_1} + \overline{u_2} + \overline{u_3} + \overline{u_4} + \overline{u_5} + \overline{u_6} + \overline{u_7} \equiv 0 \pmod 4$$

Why does it work?

- m is **even**

$$\underbrace{a x_1}_{u_1}\ \underbrace{x_2 x_3}_{u_3}\ \underbrace{x_4 x_5}_{u_5}\ \underbrace{x_6 a}_{u_7}\ \underbrace{x_1 x_2}_{u_2}\ \underbrace{x_3 x_4}_{u_4}\ \underbrace{x_5 x_6}_{u_6}\ a \qquad \underline{\text{or}}$$

$$a\ \underbrace{x_1 x_2}_{u_2}\ \underbrace{x_3 x_4}_{u_4}\ \underbrace{x_5 x_6}_{u_6}\ \underbrace{a x_1}_{u_1}\ \underbrace{x_2 x_3}_{u_3}\ \underbrace{x_4 x_5}_{u_5}\ \underbrace{x_6 a}_{u_7}$$

$$a \xrightarrow{u_1} x_1 \xrightarrow{u_2} x_2 \xrightarrow{u_3} x_3 \xrightarrow{u_4} x_4 \xrightarrow{u_5} x_5 \xrightarrow{u_6} x_6 \xrightarrow{u_7} a$$

$$\mathrm{a} \longrightarrow \{0 \xrightarrow{\bar{\mathrm{a}}} 1, 1 \xrightarrow{\bar{\mathrm{a}}} 2, 2 \xrightarrow{\bar{\mathrm{a}}} 3, 3 \xrightarrow{\bar{\mathrm{a}}} 0\} \quad \Rightarrow \quad \bar{\mathrm{a}} \equiv +1 \pmod 4$$

$$\mathrm{b} \longrightarrow \{0 \xrightarrow{\bar{\mathrm{b}}} 2, 1 \xrightarrow{\bar{\mathrm{b}}} 3, 2 \xrightarrow{\bar{\mathrm{b}}} 0, 3 \xrightarrow{\bar{\mathrm{b}}} 1\} \quad \Rightarrow \quad \bar{\mathrm{b}} \equiv +2 \pmod 4$$

$$\mathrm{c} \longrightarrow \{0 \xrightarrow{\bar{\mathrm{c}}} 3, 1 \xrightarrow{\bar{\mathrm{c}}} 0, 2 \xrightarrow{\bar{\mathrm{c}}} 1, 3 \xrightarrow{\bar{\mathrm{c}}} 2\} \quad \Rightarrow \quad \bar{\mathrm{c}} \equiv +3 \pmod 4$$

$$value + \overline{u_1} + \overline{u_2} + \overline{u_3} + \overline{u_4} + \overline{u_5} + \overline{u_6} + \overline{u_7} \equiv value \pmod 4$$

$$\overline{u_1} + \overline{u_2} + \overline{u_3} + \overline{u_4} + \overline{u_5} + \overline{u_6} + \overline{u_7} \equiv 0 \pmod 4$$

Why does it work?

- m is **even**

$$\underbrace{a x_1}_{u_1}\underbrace{x_2 x_3}_{u_3}\underbrace{x_4 x_5}_{u_5}\underbrace{x_6 a}_{u_7}\underbrace{x_1 x_2}_{u_2}\underbrace{x_3 x_4}_{u_4}\underbrace{x_5 x_6}_{u_6}\, a \qquad \underline{\text{or}}$$

$$a\,\underbrace{x_1 x_2}_{u_2}\underbrace{x_3 x_4}_{u_4}\underbrace{x_5 x_6}_{u_6}\underbrace{a x_1}_{u_1}\underbrace{x_2 x_3}_{u_3}\underbrace{x_4 x_5}_{u_5}\underbrace{x_6 a}_{u_7}$$

$$a \xrightarrow{u_1} x_1 \xrightarrow{u_2} x_2 \xrightarrow{u_3} x_3 \xrightarrow{u_4} x_4 \xrightarrow{u_5} x_5 \xrightarrow{u_6} x_6 \xrightarrow{u_7} a$$

$$\mathrm{a} \longrightarrow \{0 \xrightarrow{\bar{\mathrm{a}}} 1, 1 \xrightarrow{\bar{\mathrm{a}}} 2, 2 \xrightarrow{\bar{\mathrm{a}}} 3, 3 \xrightarrow{\bar{\mathrm{a}}} 0\} \quad \Rightarrow \quad \bar{\mathrm{a}} \equiv +1 \pmod 4$$

$$\mathrm{b} \longrightarrow \{0 \xrightarrow{\bar{\mathrm{b}}} 2, 1 \xrightarrow{\bar{\mathrm{b}}} 3, 2 \xrightarrow{\bar{\mathrm{b}}} 0, 3 \xrightarrow{\bar{\mathrm{b}}} 1\} \quad \Rightarrow \quad \bar{\mathrm{b}} \equiv +2 \pmod 4$$

$$\mathrm{c} \longrightarrow \{0 \xrightarrow{\bar{\mathrm{c}}} 3, 1 \xrightarrow{\bar{\mathrm{c}}} 0, 2 \xrightarrow{\bar{\mathrm{c}}} 1, 3 \xrightarrow{\bar{\mathrm{c}}} 2\} \quad \Rightarrow \quad \bar{\mathrm{c}} \equiv +3 \pmod 4$$

$$value + \overline{u_1} + \overline{u_2} + \overline{u_3} + \overline{u_4} + \overline{u_5} + \overline{u_6} + \overline{u_7} \equiv value \pmod 4$$

$$\overline{u_1} + \overline{u_2} + \overline{u_3} + \overline{u_4} + \overline{u_5} + \overline{u_6} + \overline{u_7} \equiv 0 \pmod 4$$

Why does it work?

- m is **even**

$$\underbrace{a x_1}_{u_1}\underbrace{x_2 x_3}_{u_3}\underbrace{x_4 x_5}_{u_5}\underbrace{x_6 a}_{u_7}\underbrace{x_1 x_2}_{u_2}\underbrace{x_3 x_4}_{u_4}\underbrace{x_5 x_6}_{u_6} a \qquad \underline{\text{or}}$$

$$a \underbrace{x_1 x_2}_{u_2}\underbrace{x_3 x_4}_{u_4}\underbrace{x_5 x_6}_{u_6}\underbrace{a x_1}_{u_1}\underbrace{x_2 x_3}_{u_3}\underbrace{x_4 x_5}_{u_5}\underbrace{x_6 a}_{u_7}$$

$$a \xrightarrow{u_1} x_1 \xrightarrow{u_2} x_2 \xrightarrow{u_3} x_3 \xrightarrow{u_4} x_4 \xrightarrow{u_5} x_5 \xrightarrow{u_6} x_6 \xrightarrow{u_7} a$$

$$a \longrightarrow \{0 \xrightarrow{\bar{a}} 1, 1 \xrightarrow{\bar{a}} 2, 2 \xrightarrow{\bar{a}} 3, 3 \xrightarrow{\bar{a}} 0\} \quad \Rightarrow \quad \bar{a} \equiv +1 \pmod 4$$

$$b \longrightarrow \{0 \xrightarrow{\bar{b}} 2, 1 \xrightarrow{\bar{b}} 3, 2 \xrightarrow{\bar{b}} 0, 3 \xrightarrow{\bar{b}} 1\} \quad \Rightarrow \quad \bar{b} \equiv +2 \pmod 4$$

$$c \longrightarrow \{0 \xrightarrow{\bar{c}} 3, 1 \xrightarrow{\bar{c}} 0, 2 \xrightarrow{\bar{c}} 1, 3 \xrightarrow{\bar{c}} 2\} \quad \Rightarrow \quad \bar{c} \equiv +3 \pmod 4$$

$$value + \overline{u_1} + \overline{u_2} + \overline{u_3} + \overline{u_4} + \overline{u_5} + \overline{u_6} + \overline{u_7} \equiv value \pmod 4$$

$$\overline{u_1} + \overline{u_2} + \overline{u_3} + \overline{u_4} + \overline{u_5} + \overline{u_6} + \overline{u_7} \equiv 0 \pmod 4$$

THEORETICAL COMPUTER SCIENCE Jagiellonian University

Why does it work?

- m is **even**

$$\underbrace{a x_1}_{u_1}\underbrace{x_2 x_3}_{u_3}\underbrace{x_4 x_5}_{u_5}\underbrace{x_6 a}_{u_7}\underbrace{x_1 x_2}_{u_2}\underbrace{x_3 x_4}_{u_4}\underbrace{x_5 x_6}_{u_6} a$$

<u>or</u>

$$a \underbrace{x_1 x_2}_{u_2}\underbrace{x_3 x_4}_{u_4}\underbrace{x_5 x_6}_{u_6}\underbrace{a x_1}_{u_1}\underbrace{x_2 x_3}_{u_3}\underbrace{x_4 x_5}_{u_5}\underbrace{x_6 a}_{u_7}$$

$$a \xrightarrow{u_1} x_1 \xrightarrow{u_2} x_2 \xrightarrow{u_3} x_3 \xrightarrow{u_4} x_4 \xrightarrow{u_5} x_5 \xrightarrow{u_6} x_6 \xrightarrow{u_7} a$$

$$\mathtt{a} \longrightarrow \{0 \xrightarrow{\bar{\mathtt{a}}} 1, 1 \xrightarrow{\bar{\mathtt{a}}} 2, 2 \xrightarrow{\bar{\mathtt{a}}} 3, 3 \xrightarrow{\bar{\mathtt{a}}} 0\} \quad \Rightarrow \quad \bar{\mathtt{a}} \equiv +1 \pmod 4$$

$$\mathtt{b} \longrightarrow \{0 \xrightarrow{\bar{\mathtt{b}}} 2, 1 \xrightarrow{\bar{\mathtt{b}}} 3, 2 \xrightarrow{\bar{\mathtt{b}}} 0, 3 \xrightarrow{\bar{\mathtt{b}}} 1\} \quad \Rightarrow \quad \bar{\mathtt{b}} \equiv +2 \pmod 4$$

$$\mathtt{c} \longrightarrow \{0 \xrightarrow{\bar{\mathtt{c}}} 3, 1 \xrightarrow{\bar{\mathtt{c}}} 0, 2 \xrightarrow{\bar{\mathtt{c}}} 1, 3 \xrightarrow{\bar{\mathtt{c}}} 2\} \quad \Rightarrow \quad \bar{\mathtt{c}} \equiv +3 \pmod 4$$

$$value + \overline{u_1} + \overline{u_2} + \overline{u_3} + \overline{u_4} + \overline{u_5} + \overline{u_6} + \overline{u_7} \equiv value \pmod 4$$

$$\overline{u_1} + \overline{u_2} + \overline{u_3} + \overline{u_4} + \overline{u_5} + \overline{u_6} + \overline{u_7} \equiv 0 \pmod 4$$

Why does it work?

- *m* is **even**

$$\underbrace{a x_1}_{u_1}\underbrace{x_2 x_3}_{u_3}\underbrace{x_4 x_5}_{u_5}\underbrace{x_6 a}_{u_7}\underbrace{x_1 x_2}_{u_2}\underbrace{x_3 x_4}_{u_4}\underbrace{x_5 x_6}_{u_6} a \qquad \underline{\text{or}}$$

$$a\underbrace{x_1 x_2}_{u_2}\underbrace{x_3 x_4}_{u_4}\underbrace{x_5 x_6}_{u_6}\underbrace{a x_1}_{u_1}\underbrace{x_2 x_3}_{u_3}\underbrace{x_4 x_5}_{u_5}\underbrace{x_6 a}_{u_7}$$

$$a \xrightarrow{u_1} x_1 \xrightarrow{u_2} x_2 \xrightarrow{u_3} x_3 \xrightarrow{u_4} x_4 \xrightarrow{u_5} x_5 \xrightarrow{u_6} x_6 \xrightarrow{u_7} a$$

$$\mathtt{a} \longrightarrow \{0 \xrightarrow{\bar{\mathtt{a}}} 1, 1 \xrightarrow{\bar{\mathtt{a}}} 2, 2 \xrightarrow{\bar{\mathtt{a}}} 3, 3 \xrightarrow{\bar{\mathtt{a}}} 0\} \quad \Rightarrow \quad \bar{\mathtt{a}} \equiv +1 \pmod 4$$

$$\mathtt{b} \longrightarrow \{0 \xrightarrow{\bar{\mathtt{b}}} 2, 1 \xrightarrow{\bar{\mathtt{b}}} 3, 2 \xrightarrow{\bar{\mathtt{b}}} 0, 3 \xrightarrow{\bar{\mathtt{b}}} 1\} \quad \Rightarrow \quad \bar{\mathtt{b}} \equiv +2 \pmod 4$$

$$\mathtt{c} \longrightarrow \{0 \xrightarrow{\bar{\mathtt{c}}} 3, 1 \xrightarrow{\bar{\mathtt{c}}} 0, 2 \xrightarrow{\bar{\mathtt{c}}} 1, 3 \xrightarrow{\bar{\mathtt{c}}} 2\} \quad \Rightarrow \quad \bar{\mathtt{c}} \equiv +3 \pmod 4$$

$$value + \overline{u_1} + \overline{u_2} + \overline{u_3} + \overline{u_4} + \overline{u_5} + \overline{u_6} + \overline{u_7} \equiv value \pmod 4$$

$$\overline{u_1} + \overline{u_2} + \overline{u_3} + \overline{u_4} + \overline{u_5} + \overline{u_6} + \overline{u_7} \equiv 0 \pmod 4$$

Why does it work?

- m is **even**

$$\underbrace{a x_1}_{u_1}\underbrace{x_2 x_3}_{u_3}\underbrace{x_4 x_5}_{u_5}\underbrace{x_6 a}_{u_7}\underbrace{x_1 x_2}_{u_2}\underbrace{x_3 x_4}_{u_4}\underbrace{x_5 x_6}_{u_6} a \qquad \underline{\text{or}}$$

$$a \underbrace{x_1 x_2}_{u_2}\underbrace{x_3 x_4}_{u_4}\underbrace{x_5 x_6}_{u_6}\underbrace{a x_1}_{u_1}\underbrace{x_2 x_3}_{u_3}\underbrace{x_4 x_5}_{u_5}\underbrace{x_6 a}_{u_7}$$

$$a \xrightarrow{u_1} x_1 \xrightarrow{u_2} x_2 \xrightarrow{u_3} x_3 \xrightarrow{u_4} x_4 \xrightarrow{u_5} x_5 \xrightarrow{u_6} x_6 \xrightarrow{u_7} a$$

$$\mathtt{a} \longrightarrow \{0 \xrightarrow{\bar{\mathtt{a}}} 1, 1 \xrightarrow{\bar{\mathtt{a}}} 2, 2 \xrightarrow{\bar{\mathtt{a}}} 3, 3 \xrightarrow{\bar{\mathtt{a}}} 0\} \quad \Rightarrow \quad \bar{\mathtt{a}} \equiv +1 \pmod 4$$

$$\mathtt{b} \longrightarrow \{0 \xrightarrow{\bar{\mathtt{b}}} 2, 1 \xrightarrow{\bar{\mathtt{b}}} 3, 2 \xrightarrow{\bar{\mathtt{b}}} 0, 3 \xrightarrow{\bar{\mathtt{b}}} 1\} \quad \Rightarrow \quad \bar{\mathtt{b}} \equiv +2 \pmod 4$$

$$\mathtt{c} \longrightarrow \{0 \xrightarrow{\bar{\mathtt{c}}} 3, 1 \xrightarrow{\bar{\mathtt{c}}} 0, 2 \xrightarrow{\bar{\mathtt{c}}} 1, 3 \xrightarrow{\bar{\mathtt{c}}} 2\} \quad \Rightarrow \quad \bar{\mathtt{c}} \equiv +3 \pmod 4$$

$$value + \overline{u_1} + \overline{u_2} + \overline{u_3} + \overline{u_4} + \overline{u_5} + \overline{u_6} + \overline{u_7} \equiv value \pmod 4$$

$$\overline{u_1} + \overline{u_2} + \overline{u_3} + \overline{u_4} + \overline{u_5} + \overline{u_6} + \overline{u_7} \equiv 0 \pmod 4$$

Why does it work?

- m is **even**

$$\underbrace{a x_1}_{u_1}\ \underbrace{x_2 x_3}_{u_3}\ \underbrace{x_4 x_5}_{u_5}\ \underbrace{x_6 a}_{u_7}\ \underbrace{x_1 x_2}_{u_2}\ \underbrace{x_3 x_4}_{u_4}\ \underbrace{x_5 x_6}_{u_6}\ a$$

or

$$a\ \underbrace{x_1 x_2}_{u_2}\ \underbrace{x_3 x_4}_{u_4}\ \underbrace{x_5 x_6}_{u_6}\ \underbrace{a x_1}_{u_1}\ \underbrace{x_2 x_3}_{u_3}\ \underbrace{x_4 x_5}_{u_5}\ \underbrace{x_6 a}_{u_7}$$

$$\overline{u_1} + \overline{u_2} + \overline{u_3} + \overline{u_4} + \overline{u_5} + \overline{u_6} + \overline{u_7} \equiv 0 \pmod 4$$

Lemma

If w is a subword of t, then $||w|_a - |w|_c| \leq 1$, where $|w|_s$ denotes the number of times the letter s appears in w.

By lemma and counting the numbers of a, b, c in u:

$$\overline{u_1} + \overline{u_2} + \overline{u_3} + \overline{u_4} + \overline{u_5} + \overline{u_6} + \overline{u_7} \not\equiv 0 \pmod 4$$

$\Rightarrow$ contradiction

THEORETICAL COMPUTER SCIENCE Jagiellonian University

Why does it work?

- m is **even**

$$\underbrace{a x_1}_{u_1}\ \underbrace{x_2 x_3}_{u_3}\ \underbrace{x_4 x_5}_{u_5}\ \underbrace{x_6 a}_{u_7}\ \underbrace{x_1 x_2}_{u_2}\ \underbrace{x_3 x_4}_{u_4}\ \underbrace{x_5 x_6}_{u_6}\ a \qquad \underline{\text{or}}$$

$$a\ \underbrace{x_1 x_2}_{u_2}\ \underbrace{x_3 x_4}_{u_4}\ \underbrace{x_5 x_6}_{u_6}\ \underbrace{a x_1}_{u_1}\ \underbrace{x_2 x_3}_{u_3}\ \underbrace{x_4 x_5}_{u_5}\ \underbrace{x_6 a}_{u_7}$$

$$\overline{u_1} + \overline{u_2} + \overline{u_3} + \overline{u_4} + \overline{u_5} + \overline{u_6} + \overline{u_7} \equiv 0 \pmod 4)$$

Lemma

If w is a subword of t, then $\left||w|_{\mathtt{a}} - |w|_{\mathtt{c}}\right| \leq 1$, where $|w|_s$ denotes the number of times the letter s appears in w.

By lemma <u>and</u> counting the numbers of a, b, c in u:

$$\overline{u_1} + \overline{u_2} + \overline{u_3} + \overline{u_4} + \overline{u_5} + \overline{u_6} + \overline{u_7} \not\equiv 0 \pmod 4)$$

$\Rightarrow$ contradiction

Why does it work?

- m is **even**

$$\underbrace{a x_1}_{u_1}\ \underbrace{x_2 x_3}_{u_3}\ \underbrace{x_4 x_5}_{u_5}\ \underbrace{x_6 a}_{u_7}\ \underbrace{x_1 x_2}_{u_2}\ \underbrace{x_3 x_4}_{u_4}\ \underbrace{x_5 x_6}_{u_6}\ a \qquad \underline{\text{or}}$$

$$a\ \underbrace{x_1 x_2}_{u_2}\ \underbrace{x_3 x_4}_{u_4}\ \underbrace{x_5 x_6}_{u_6}\ \underbrace{a x_1}_{u_1}\ \underbrace{x_2 x_3}_{u_3}\ \underbrace{x_4 x_5}_{u_5}\ \underbrace{x_6 a}_{u_7}$$

$$\overline{u_1} + \overline{u_2} + \overline{u_3} + \overline{u_4} + \overline{u_5} + \overline{u_6} + \overline{u_7} \equiv 0 \pmod 4$$

Lemma

If w is a subword of t, then $||w|_{\mathtt{a}} - |w|_{\mathtt{c}}| \leq 1$, where $|w|_s$ denotes the number of times the letter s appears in w.

By lemma and counting the numbers of a, b, c in u:

$$\overline{u_1} + \overline{u_2} + \overline{u_3} + \overline{u_4} + \overline{u_5} + \overline{u_6} + \overline{u_7} \not\equiv 0 \pmod 4$$

$\Rightarrow$ contradiction

Why does it work?

- m is **even**

$$\underbrace{a x_1}_{u_1}\ \underbrace{x_2 x_3}_{u_3}\ \underbrace{x_4 x_5}_{u_5}\ \underbrace{x_6 a}_{u_7}\ \underbrace{x_1 x_2}_{u_2}\ \underbrace{x_3 x_4}_{u_4}\ \underbrace{x_5 x_6}_{u_6}\ a \qquad \underline{\text{or}}$$

$$a\ \underbrace{x_1 x_2}_{u_2}\ \underbrace{x_3 x_4}_{u_4}\ \underbrace{x_5 x_6}_{u_6}\ \underbrace{a x_1}_{u_1}\ \underbrace{x_2 x_3}_{u_3}\ \underbrace{x_4 x_5}_{u_5}\ \underbrace{x_6 a}_{u_7}$$

$$\overline{u_1} + \overline{u_2} + \overline{u_3} + \overline{u_4} + \overline{u_5} + \overline{u_6} + \overline{u_7} \equiv 0 \pmod 4$$

Lemma

If w is a subword of t, then $\left||w|_{\mathtt{a}} - |w|_{\mathtt{c}}\right| \leq 1$, where $|w|_s$ denotes the number of times the letter s appears in w.

By lemma <u>and</u> counting the numbers of a, b, c in u:

$$\overline{u_1} + \overline{u_2} + \overline{u_3} + \overline{u_4} + \overline{u_5} + \overline{u_6} + \overline{u_7} \not\equiv 0 \pmod 4$$

$\Rightarrow$ contradiction

Why does it work?

- m is **even**

$$\underbrace{{\color{red}a}x_1}_{u_1}\underbrace{x_2x_3}_{u_3}\underbrace{x_4x_5}_{u_5}\underbrace{x_6{\color{red}a}}_{u_7}\underbrace{x_1x_2}_{u_2}\underbrace{x_3x_4}_{u_4}\underbrace{x_5x_6}_{u_6}{\color{red}a} \qquad \text{or}$$

$${\color{red}a}\underbrace{x_1x_2}_{u_2}\underbrace{x_3x_4}_{u_4}\underbrace{x_5x_6}_{u_6}\underbrace{{\color{red}a}x_1}_{u_1}\underbrace{x_2x_3}_{u_3}\underbrace{x_4x_5}_{u_5}\underbrace{x_6{\color{red}a}}_{u_7}$$

$$\overline{u_1} + \overline{u_2} + \overline{u_3} + \overline{u_4} + \overline{u_5} + \overline{u_6} + \overline{u_7} \equiv 0 \pmod 4$$

Lemma

If w is a subword of t, then $\left||w|_{\mathtt{a}} - |w|_{\mathtt{c}}\right| \leq 1$, where $|w|_s$ denotes the number of times the letter s appears in w.

By lemma and counting the numbers of a, b, c in u:

$$\overline{u_1} + \overline{u_2} + \overline{u_3} + \overline{u_4} + \overline{u_5} + \overline{u_6} + \overline{u_7} \not\equiv 0 \pmod 4$$

$\Rightarrow$ contradiction

The end

Questions?