On a local similarity of graphs

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Problem

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k-similar graphs

We say that two graphs G and H, having the same number of vertices n, are k-similar if they contain a common induced subgraph of order k. Assume that $l \geq 3$.

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$\eta(k, l)$, Dzido, Krzywdziński 2015

Let $\eta(k, l)$ be the smallest n such that in any family of l graphs on n vertices there exists a k-similar pair of graphs.

Motivation 1

Similar questions was put by Chung, Erdös and Spencer and by Chung, Erdös, Graham, Ulam and Yao. The authors of those articles were interested in finding a **common induced subgraph** of two dense graphs.

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Motivation 2

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Motivation 2

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Motivation 3

An additional motivation for studying $\eta(k, l)$ is the fact that it is closely related to the Ramsey number.

Ramsey number - definition

The Ramsey number $R(k, k)$ is the minimum number n such that any graph G on n vertices contains either a k-vertex clique K_k, or an independent set of size k denoted by $\overline{K_k}$.

Theorem 1

Let $k \geq 3$. Then

$$\eta(k, 3) = R(k, k).$$

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Theorem 2

Let $k, l \geq 3$. Then the number $\eta(k,l)$ is a well-defined finite number.

Theorem 3

Let $k \geq 3$. Then $\eta(k, 2k+1) \leq R(k-1, k-1)$.

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Corollary

$\eta(k, l)$ might be considered a non–trivial generalisation of the Ramsey number.

Theorem 4

Let $k \geq 3$. Then $\eta(k, 4) > (k-1)^2$.

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Theorem 5

Let $k,l \geq 3$ and $t \geq 1$. Then

$$\eta(tk,l) > t\eta(\lceil k/t \rceil, tl) - t.$$

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Bounds on $|Q_k|$

$\sqrt{2^{\binom{k}{2}}} < |Q_k| < 2^{\binom{k}{2}}$ by P. O. de Wet

$2^{\binom{k}{2}}$ is the number of labelled graphs on k vertices. As each such graph admits at most $k!$ automorphisms, a lower bound on $|Q_k|$ is $2^{\binom{k}{2}}/k!$, which is much stronger than the constructive bound of de Wet.

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Fact 1

$\eta(k,l) = k$ for $l > |Q_k|$ and $k \geq 3$.

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Fact 1

$\eta(k,l) = k$ for $l > |Q_k|$ and $k \geq 3$.

Fact 2

$\eta(k,l) = k+1$ for $l = |Q_k|$ and $k \geq 4$.

A graph G of order n is vertex-transitive if all its $(n-1)$-vertex subgraphs are isomorphic. We will denote by S_n the set of all vertex-transitive graphs of order n.

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Lemma

$|S_n| \leq \frac{|Q_{n-1}|}{2}$ for $n > 6$.

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Lemma

$|S_n| \leq \frac{|Q_{n-1}|}{2}$ for $n > 6$.

Theorem 6

$\eta(k, l) = k + 1$ for $k \geq 5$ and $|Q_k| \geq l > \frac{3}{4}|Q_k|$.

Theorem 7

Let $l \geq 3$. Then

$$\eta(3,l) = \begin{cases} 6 & \text{for } l = 3; \\ 5 & \text{for } l = 4; \\ 3 & \text{for } l \geq 5; \end{cases} \quad \text{and} \quad \eta(4,l) = \begin{cases} 18 & \text{for } l = 3; \\ 10 & \text{for } l = 4; \\ 7 & \text{for } l = 5; \\ 6 & \text{for } l = 6,7; \\ 5 & \text{for } 8 \leq l \leq 11; \\ 4 & \text{for } \geq 12. \end{cases}$$

More results and details in:

Article

Dzido T., Krzywdziński K.: On a local similarity of graphs, to appear in *Discrete Mathematics*

Thank you.