Smaller Selection Networks for Cardinality Constraints Encoding

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Outline

1. Motivation
2. Comparator networks
3. New selection networks
4. Conclusion

Motivation

To encode *cardinality constraints*. Examples:

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- $x_1 + x_2 + x_3 < 3$
- $y_1 + y_2 \geq 1$
- $z_1 + z_2 + z_3 + z_4 = 2$

Motivation

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- $y_1 + y_2 \geq 1$
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$x_1, \ldots, x_n \in \{0, 1\}$, $k \in \mathbb{N}$: $\quad x_1 + \ldots + x_n \# k$.

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SAT-solvers:

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$$I \rightsquigarrow \phi(I) \rightsquigarrow \text{SAT-solver}$$

$$I \rightsquigarrow \phi'(I) \wedge (\underbrace{x_1 + \ldots + x_n < k}_{C}) \rightsquigarrow \phi'(I) \wedge \psi(C) \rightsquigarrow \text{SAT-solver}$$

Example:

Motivation

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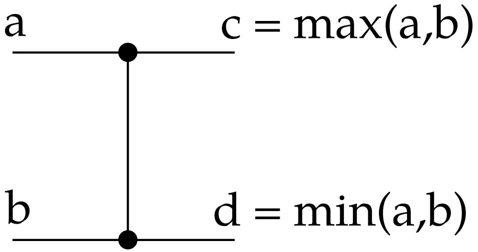
- $x_1 + x_2 + x_3 \leq 1$

Motivation

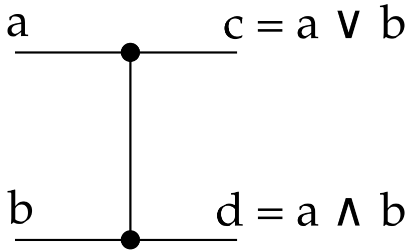
Example:

- $x_1 + x_2 + x_3 \leq 1$
- $(\bar{x}_1 \vee \bar{x}_2) \wedge (\bar{x}_1 \vee \bar{x}_3) \wedge (\bar{x}_2 \vee \bar{x}_3)$

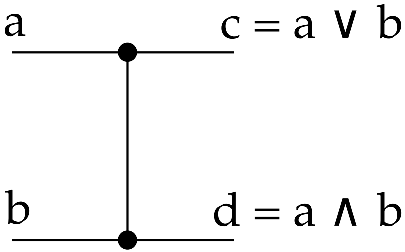
Comparator networks



Comparator networks

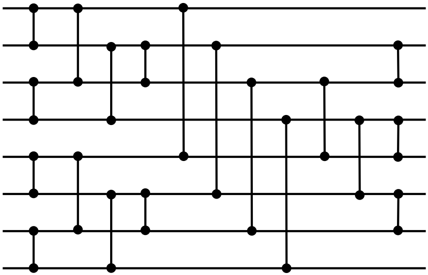


Comparator networks



$$comp(a, b, c, d) \Leftrightarrow (c \Leftrightarrow a \vee b) \wedge (d \Leftrightarrow a \wedge b)$$

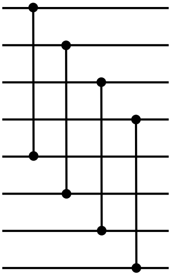
Comparator networks



Bather's odd-even sorting network (1968)

Comparator networks

Splitter:

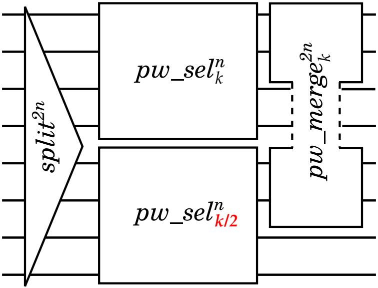


Definition (domination)

$\bar{x} \in \mathbb{N}^n$ dominates $\bar{y} \in \mathbb{N}^n$ if $x_i \geq y_i$ (for $i = 1..n$).

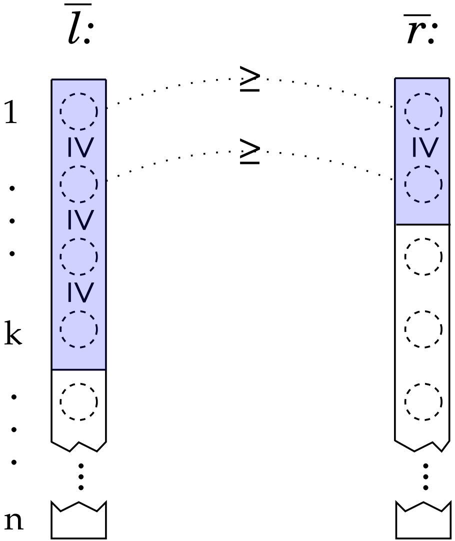
Comparator networks

Pairwise *selection*:



(Codish and Zazon-Ivry, 2012)

Comparator networks



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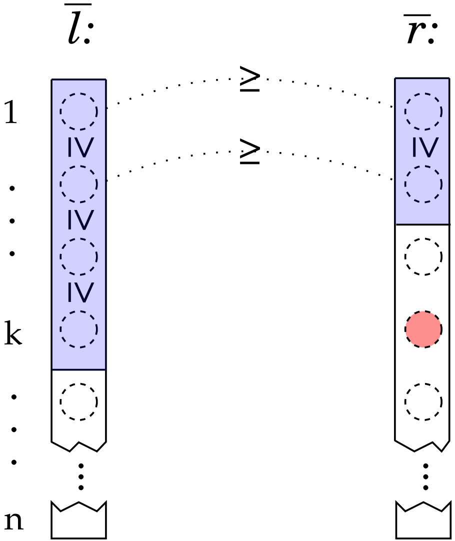
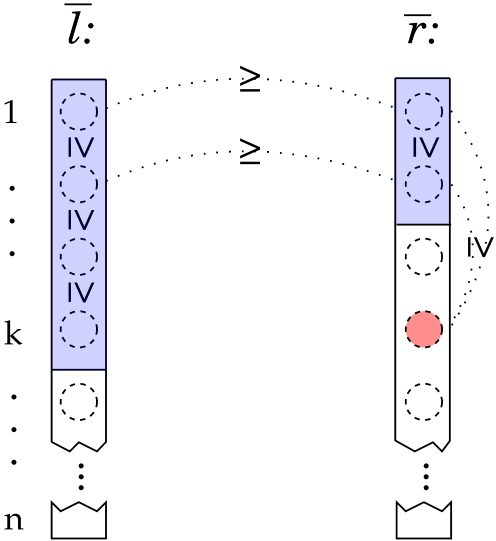


Comparator networks



New selection networks

Codish and Zazon-Ivry, 2012:

- $|pw\_merge| = k \log k - k + 1$

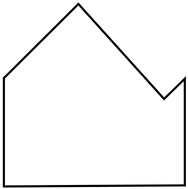
Karpiński and Piotrów, 2015:

- $|pw\_bit\_merge| = \frac{1}{2} k \log k + \frac{1}{2} k$
- $|pw\_hbit\_merge| = \frac{1}{2} k \log k$

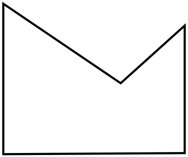
New selection networks

Definition (bitonic sequence)

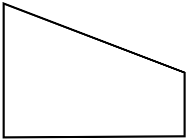
A sequence $\bar{x} \in \mathbb{N}^n$ is a bitonic sequence if $x_1 \leq \ldots \leq x_i \geq \ldots \geq x_n$ for some i, where $1 \leq i \leq n$, or a circular shift of such sequence.



(bitonic)



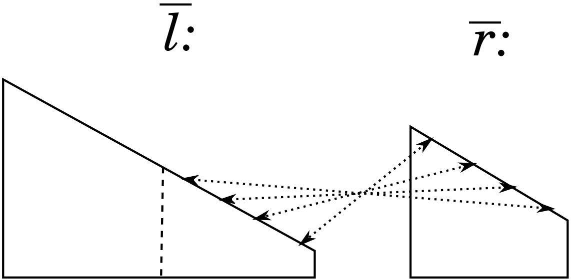
(v-shaped)



(nonincreasing)

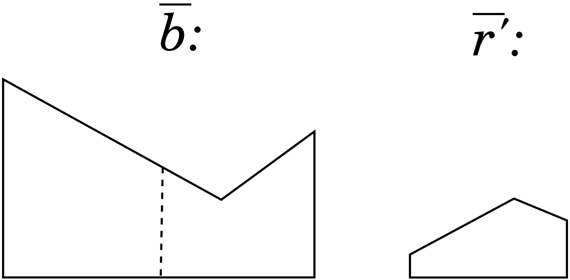
New selection networks

Step 1:



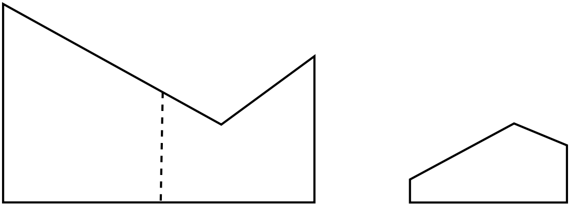
New selection networks

Step 1:



New selection networks

Step 1: $\overline{b}$: $\overline{r}'$:



Step 2: Use Batcher's bitonic merger: $bit\_merge(\bar{b})$

New selection networks

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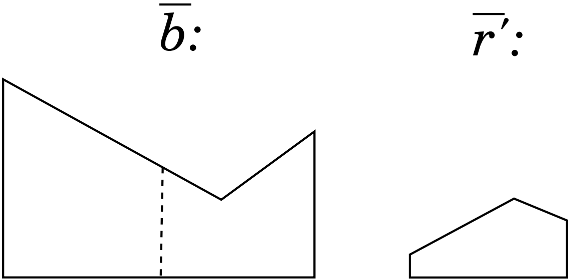

Step 2: Use Batcher's bitonic merger: $\mathit{bit\_merge}(\bar{b})$
$(|\mathit{bit\_merge}^k| = \frac{1}{2} k \log k)$

New selection networks

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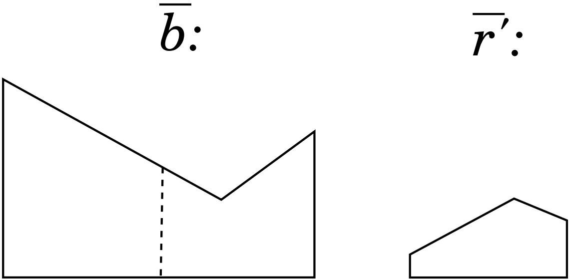

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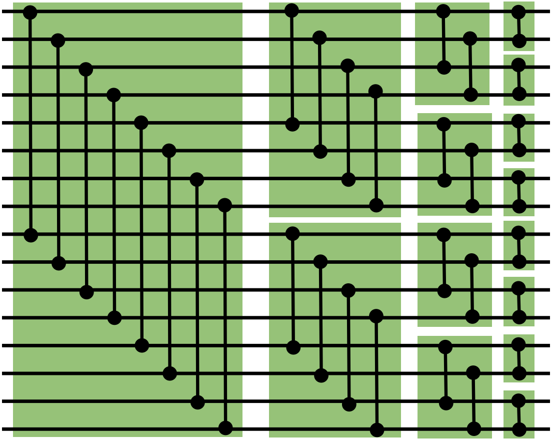
$(|bit\_merge^k| = \frac{1}{2} k \log k)$

$|pw\_bit\_merge| = \frac{1}{2} k \log k + \frac{1}{2} k$

New selection networks

bit_merge^{16}:



New selection networks

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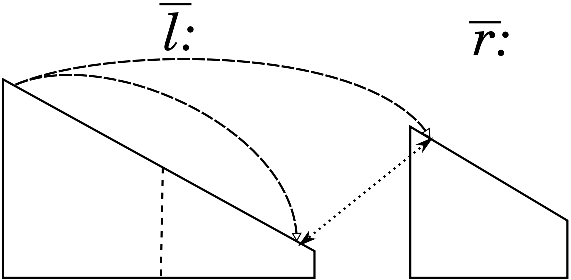
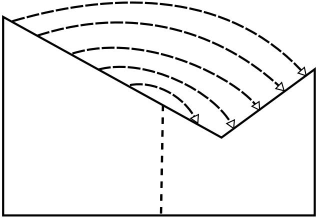
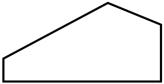


Step 1:

$\overline{b}$:



$\overline{r}'$:



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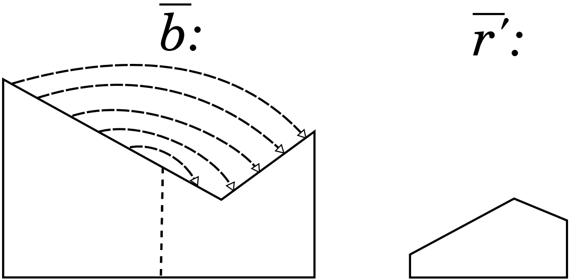

Definition (s-domination)

A sequence $\bar{b} = \langle b_1, b_2, \ldots, b_k \rangle$ is *s-dominating* if:
$\forall_{1 \le j \le k/2}\ b_j \ge b_{k-j+1}$.

New selection networks

Observation

If $\bar{b} = \langle b_1, b_2, \ldots, b_k \rangle$ is v-shaped s-dominating, then $\langle b_1, \ldots, b_{k/4} \rangle$ dominates $\langle b_{k/2+1}, \ldots, b_{3k/4} \rangle$.

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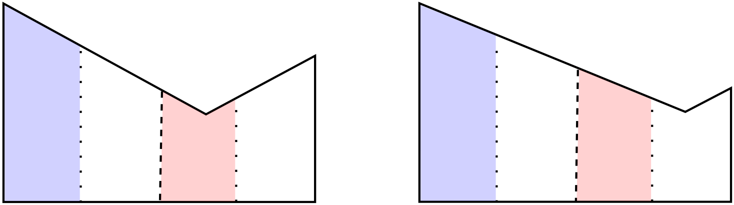

New selection networks

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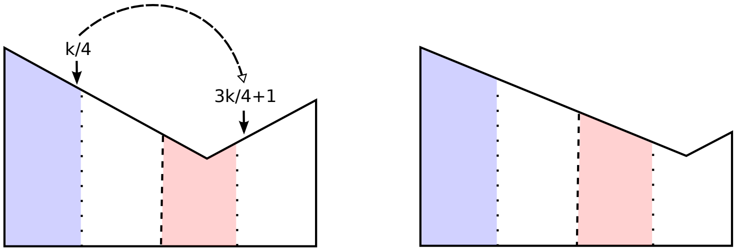
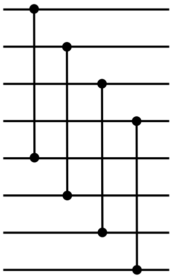


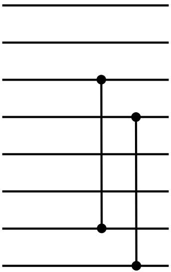
New selection networks

Splitter:

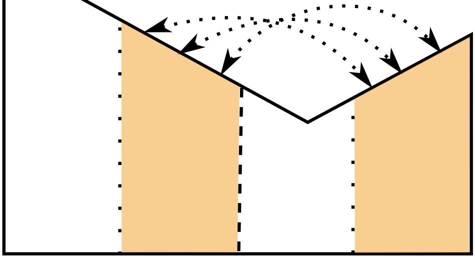


New selection networks

Half splitter:

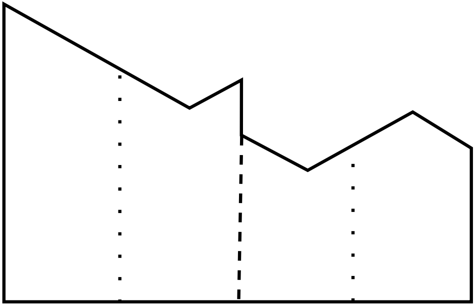


Observation:



New selection networks

Observation:



New selection networks

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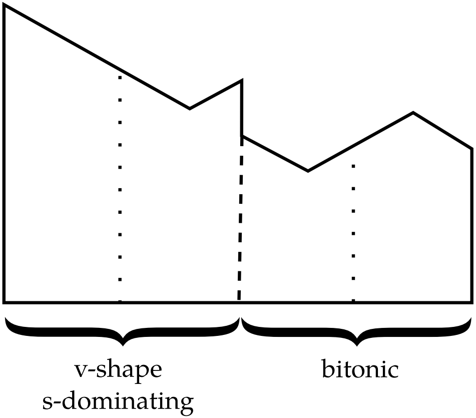
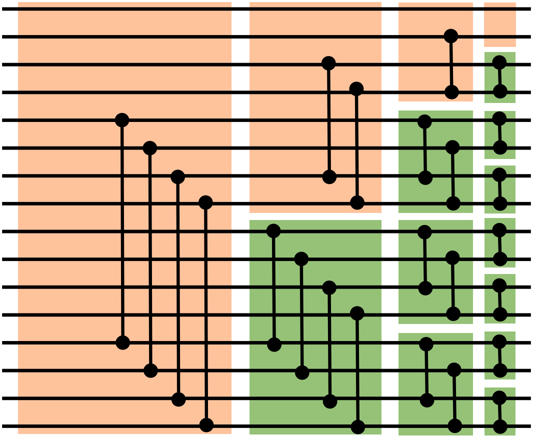
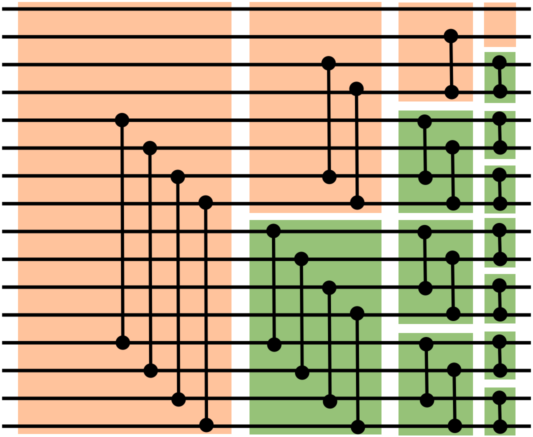


New selection networks



New selection networks



$$|pw\_hbit\_merge| = (\frac{1}{2}k\log k - \frac{1}{2}k) + \frac{1}{2}k = \frac{1}{2}k\log k$$

Conclusion

Theorem

$|pw\_hbit\_sel^n_k| \leq |pw\_sel^n_k|$

Thank You!