Dynamic algorithms for the Steiner tree problem

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joint work with
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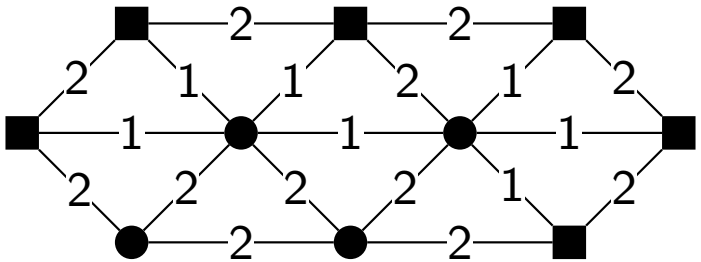
The Steiner Minimum Tree Problem

Definition (The Steiner Minimum Tree Problem (SMT))

Input: A weighted graph $G = (V, E, w)$, a terminal set $S \subseteq V$.



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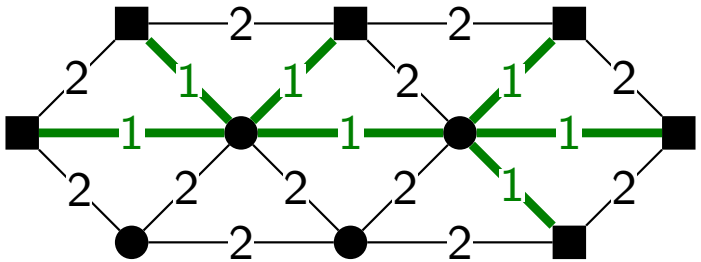
Input: A weighted graph $G = (V, E, w)$, a terminal set $S \subseteq V$.
Output: Minimum-weight tree T with $S \subseteq V(T)$.



The dynamic setting

Setting

- Initially, we get a graph $G = (V, E, w)$ whereas terminal set $S = \emptyset$.
- Then, requests arrive one by one.
- Request: `insert`(v) or `delete`(v).
  Add or delete vertex from terminal set S.
- At each step, maintain a good approximation of $\mathrm{Steiner}(G, S)$.

Efficiency measure.

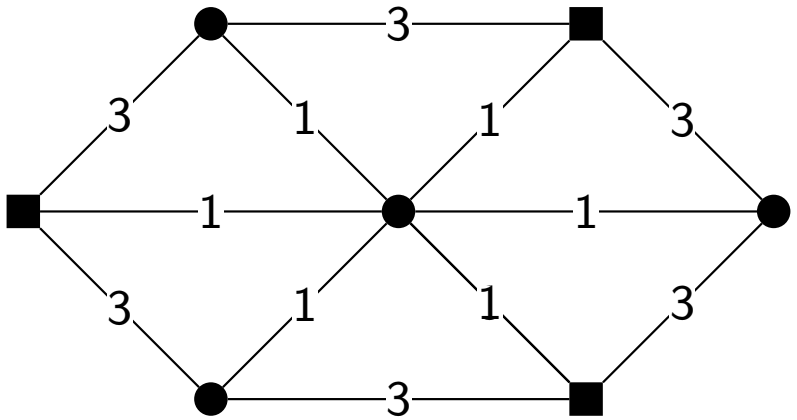
The total time for computation at each step.

Simple approximation

Definition (Metric closure)

For a weighted graph $G = (V, E, w)$,
the *metric closure* of G is a complete graph $\overline{G} = (V, \binom{V}{2}, \delta)$ with costs

$$\delta(uv) = \mathrm{dist}_G(u, v).$$



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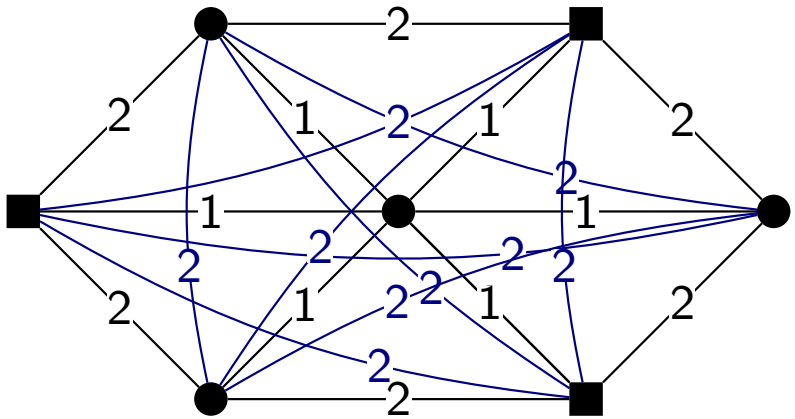

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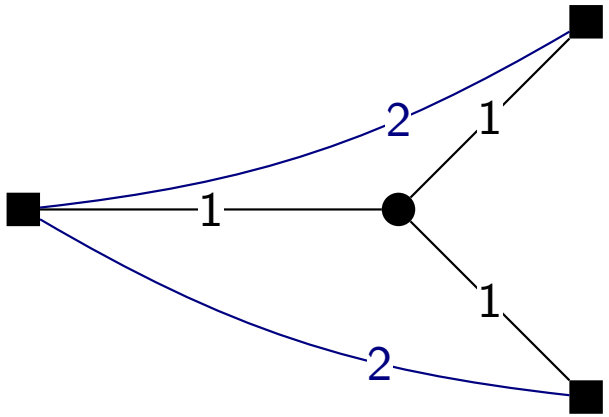

Lemma

$$\mathrm{MST}\left(\overline{G}[S]\right) \leq 2\mathrm{Steiner}(G, S).$$

Straightforward approach

Theorem (Holm et. al.)

There exists a fully dynamic MSF algorithm, that for a graph on n vertices supports m edge additions and removals in $O(m\log^4 n)$ total time.

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Approach:

- Maintain $\overline{G}[S]$ under required updates:
- `insert`(s):: add all the edges adjacent to s in $\overline{G}[S \cup s]$
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- Use the algorithm of Holm et. al. to maintain MST on $\overline{G}[S]$
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Issues:

- Minor issue: the weights of edges in the metric closure are needed
- Bigger issue: linear time per update is not satisfactory

Our results

Setting	apx	update time	preprocessing time
general, full	$8k-4$	$\tilde{O}(kn^{1/k})$	$O(kmn^{1/k})$
general, full	$6+\varepsilon$	$\tilde{O}(\varepsilon^{-5}\sqrt{n}\log D)$	$\tilde{O}(\varepsilon^{-3}n\sqrt{n}\log D)$
general, incremental	$6+\varepsilon$	$\tilde{O}(\varepsilon^{-1}\sqrt{n})$	$\tilde{O}(\varepsilon^{-1}\sqrt{n}m)$
general, decremental	$6+\varepsilon$	$O(\varepsilon^{-1}\sqrt{n}\log n)$	$\tilde{O}(\sqrt{n}m)$
planar, full	$4+\varepsilon$	$O(\varepsilon^{-1}\log^6 n)$	$O(\varepsilon^{-1}n\log^2 n)$
planar, full	$2+\varepsilon$	$\tilde{O}(\varepsilon^{-5.5}\sqrt{n}\log D)$	$\tilde{O}(\varepsilon^{-5}n\log D)$
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A more careful approach

Before:

dynamic MST algorithm applied to dynamically maintained **metric closure**

Now:

dynamic MST algorithm applied to dynamically maintained **bipartite emulator**

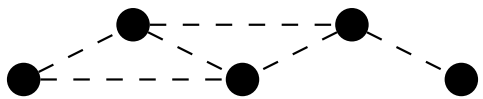
Idea:

The bipartite emulator approximates the distances in the metric closure. It is, however, sparser and hence can be modified with fewer edge modifications.

Bipartite emulators

Definition

An α-bipartite emulator of a graph $G = (V, E, w)$



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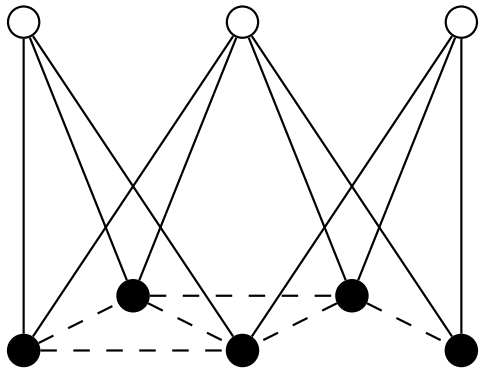

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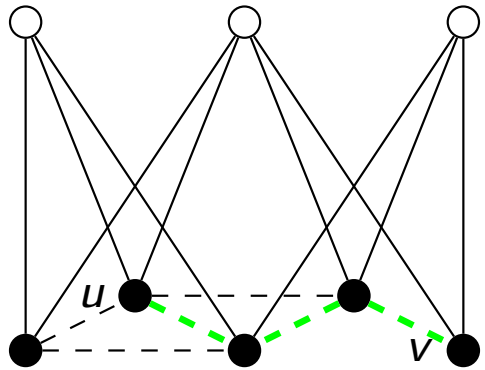

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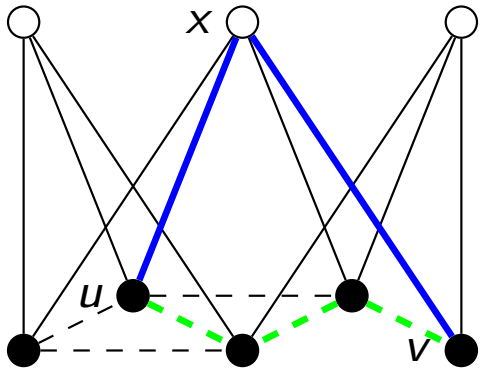
$$w'(ux) + w'(vx) \leq \alpha \cdot \mathrm{dist}_G(u, v)$$

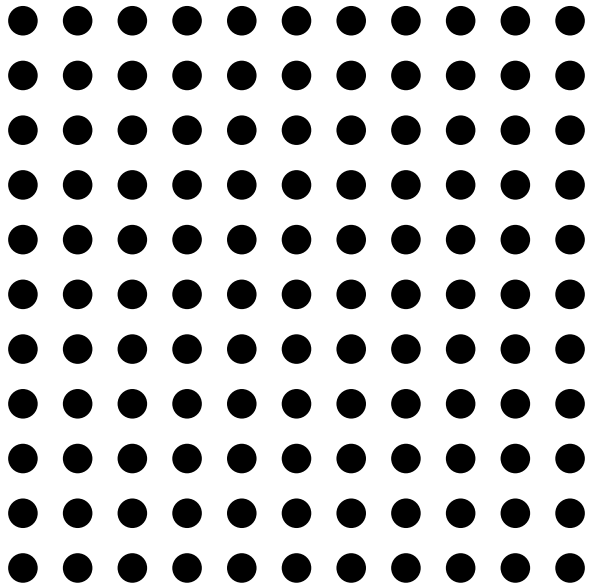


Thorup and Zwick distance oracle

Theorem (Thorup and Zwick)

There is a data structure, which
takes $O(kmn^{1/k})$ *time to* ***preprocess***,
occupies $O(kn^{1+1/k})$ ***space***, *and*
allows to estimate distances in ***time*** $O(k)$.
The ***stretch*** *of the produced estimates is bounded from above by* $2k-1$
and no distance is underestimated.

Thorup and Zwick distance oracle for $k = 2$



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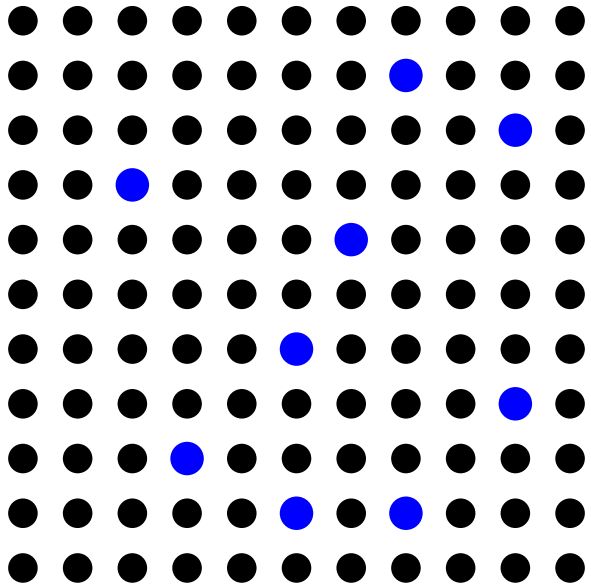

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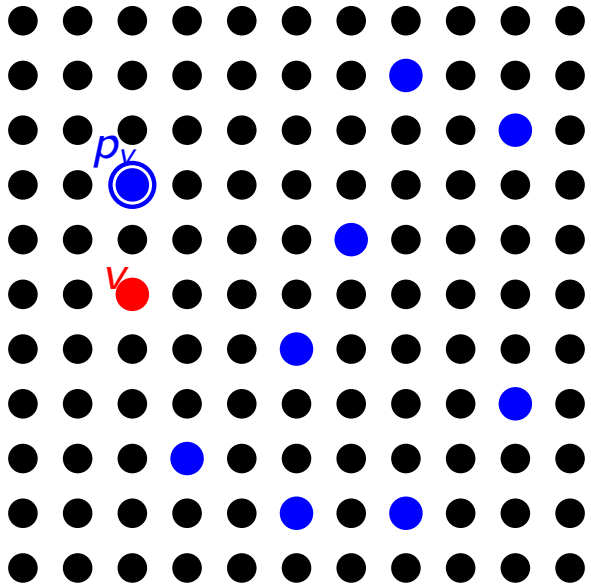

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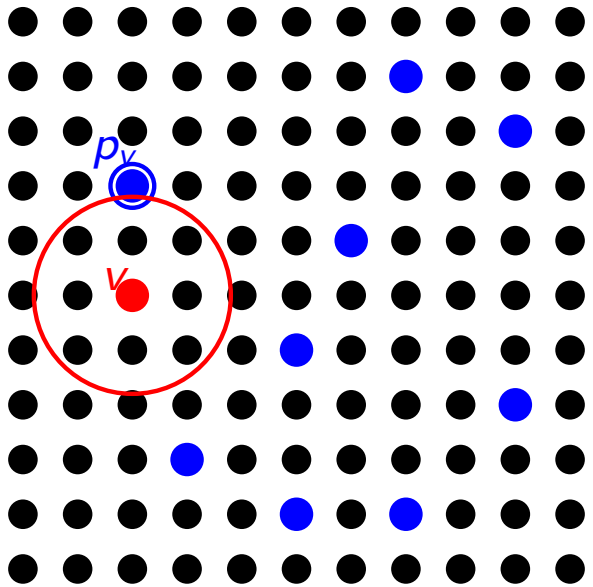

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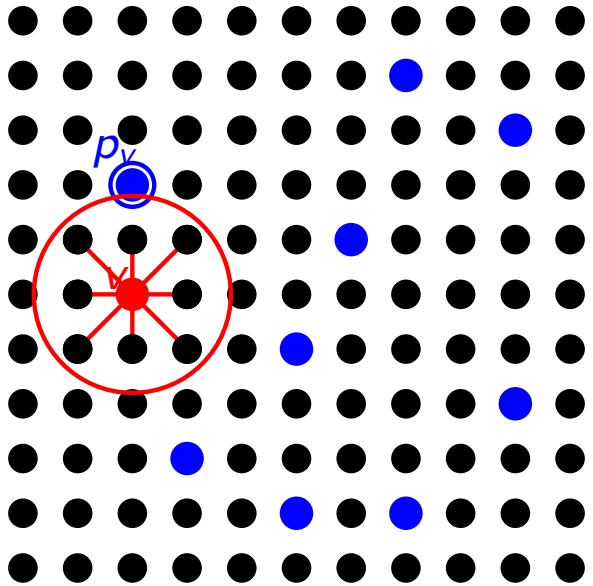
- Every $v \in V$ remembers distances to $B(v)$



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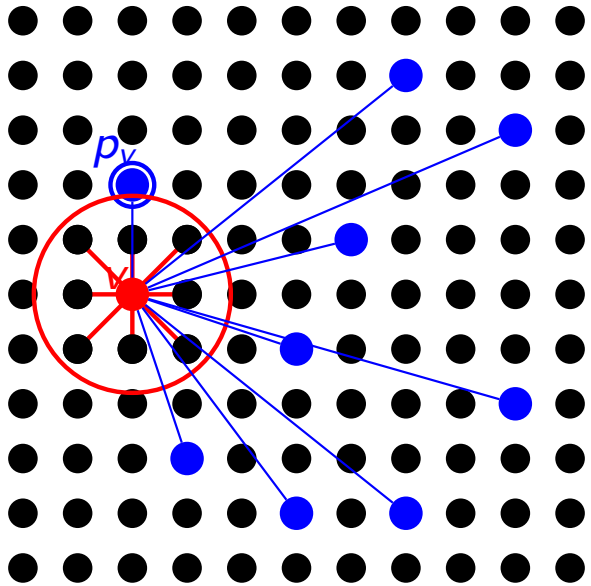

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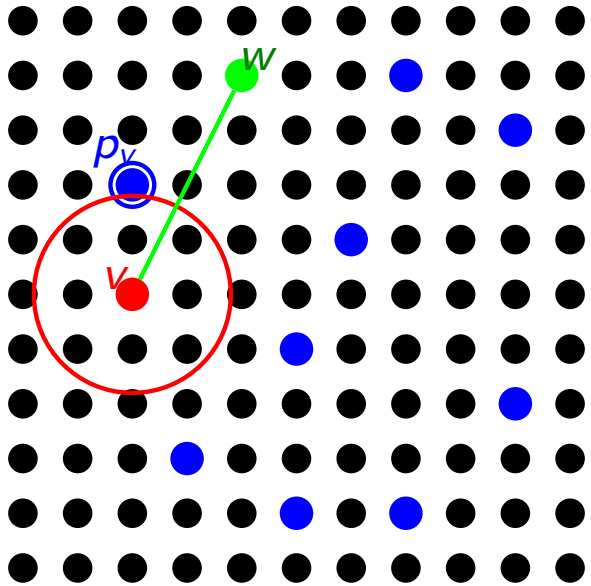

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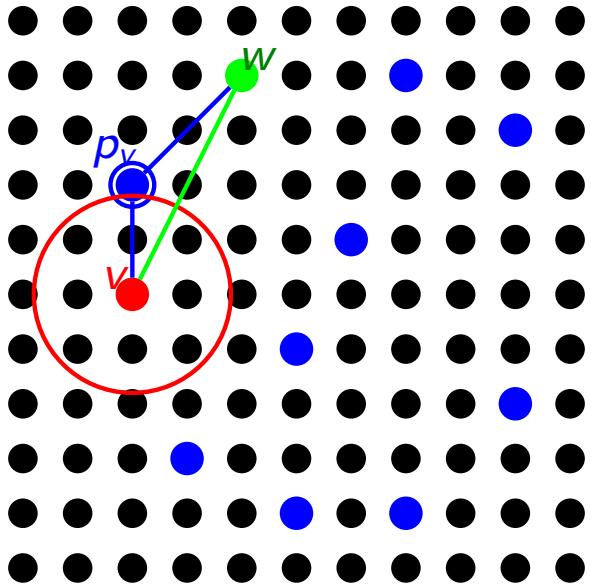
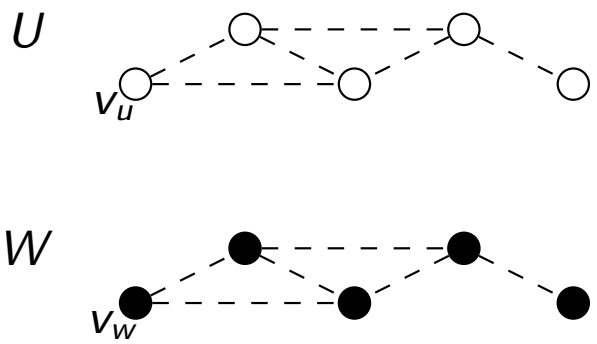


Lemma

If $w \notin B(v)$ *then* $\mathrm{dist}(v, p_v) + \mathrm{dist}(p_v, w) \leq 3\mathrm{dist}(v, w)$.

From TZ-oracle to a bipartite emulator

- make two copies of V: U and W. Now every vertex occurs in two copies: v_u and v_w.



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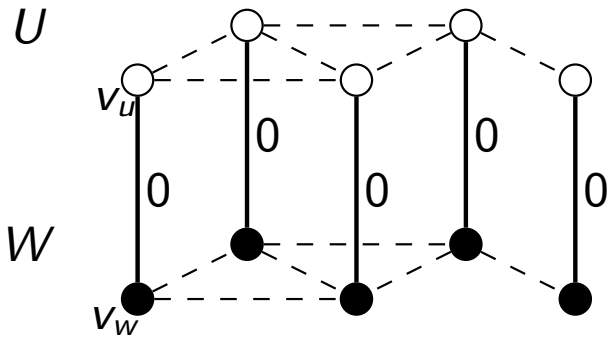

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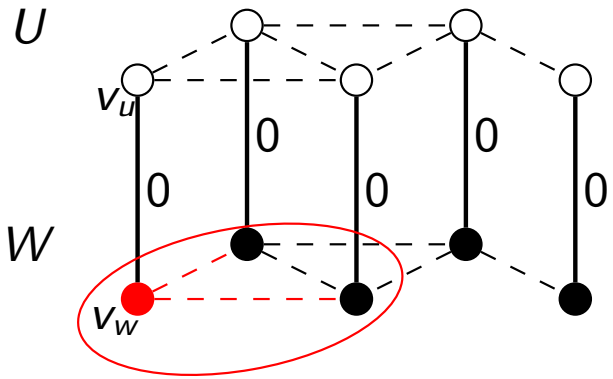

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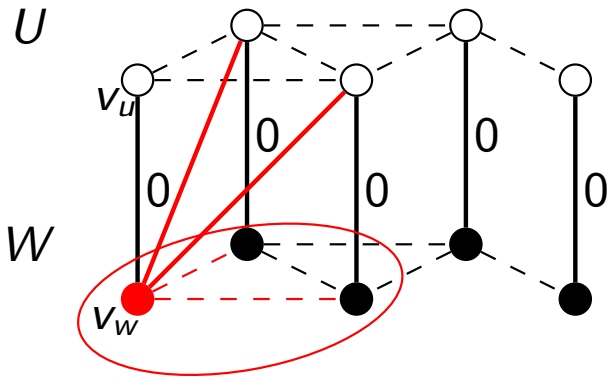

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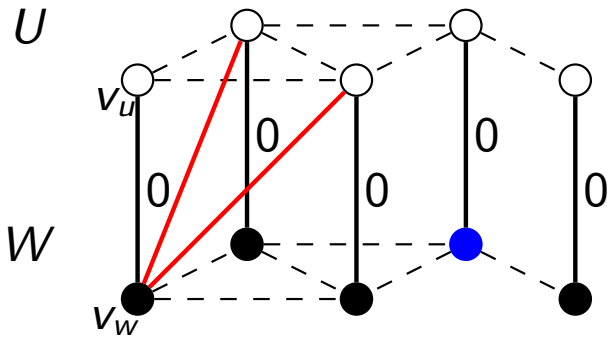

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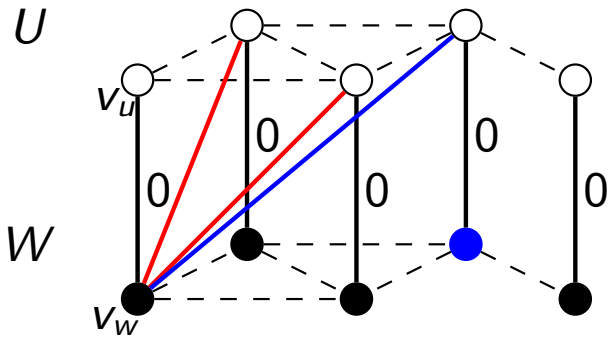

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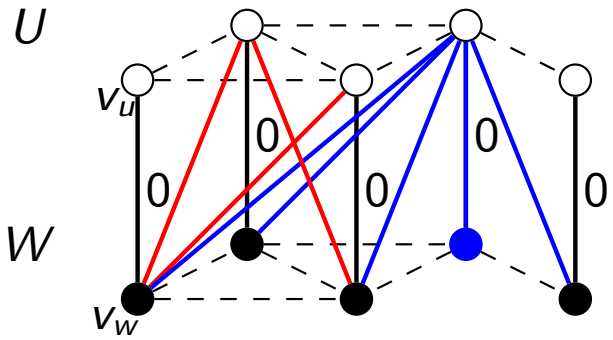

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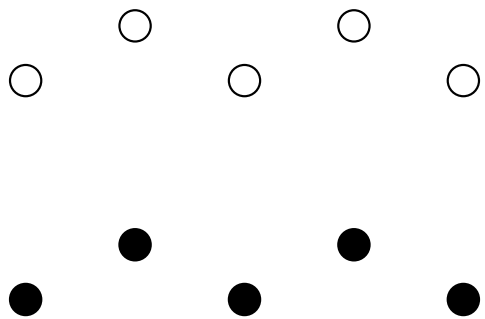

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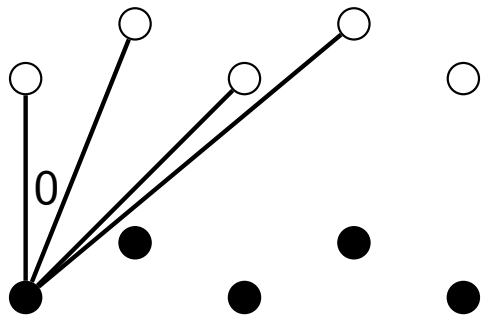

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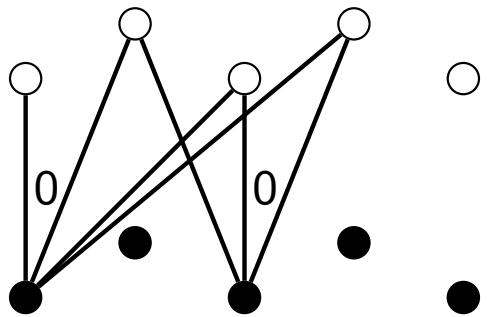

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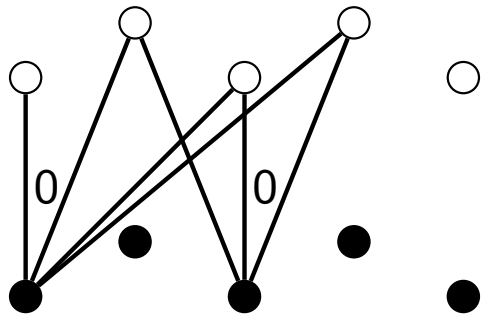

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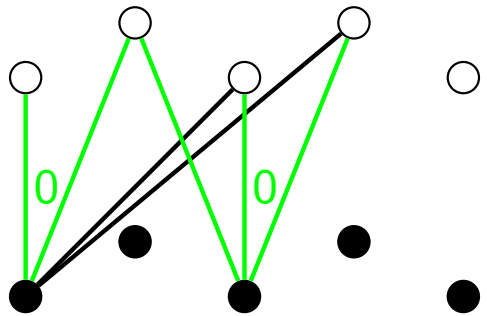

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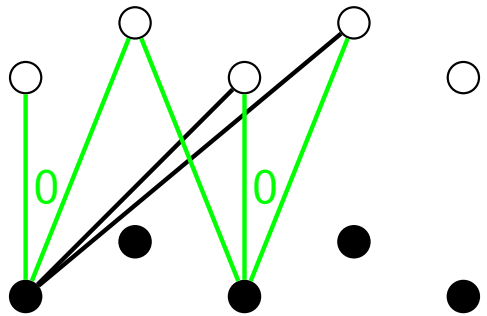
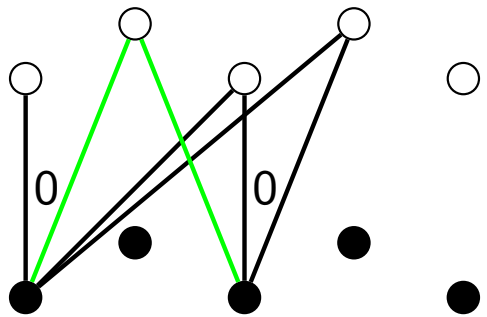
  - Maintain MST on $H(G)$ using the algorithm of Holm et. al.
- Since $|B(s) \cup P| \in O(\sqrt{n})$, updates take $O(\sqrt{n}\log^4 n)$ time



The algorithm

- Build $B(G) = (U \cup W, E, w')$-the bipartite emulator for G.
- Maintain dynamically a subgraph $H(G) \subseteq B(G)$:
  - Set $H(G) := (U \cup W, \emptyset)$
  - `insert`(s) : add all the edges adjacent to s_w to $H(G)$
  - `delete`(s) : remove all the edges adjacent to s_w from $H(G)$
  - Maintain MST on $H(G)$ using the algorithm of Holm et. al.
- Since $|B(s) \cup P| \in O(\sqrt{n})$, updates take $O(\sqrt{n}\log^4 n)$ time



Lemma

If we cut off the leaves in U from the maintained tree, we get a 2α-approximation of $MST(\overline{G}[S])$

Open ends

- In all our algorithms, for simplicity, we approximate MST rather than SMT. Can one use better approximations in the dynamic setting, and still guarantee a reasonable update time?
- For what other hard problems can one develop efficient dynamic approximation algorithms?